Frames and Symbolisms

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Chapter 1

The Need for Philosophical Tools

Many philosophers have dedicated their lives to answering questions of truth and ethics. Despite their efforts, such questions are still the source of heated debate in philosophical circles. Philosophers would be the first to admit that these debates are their bread and butter, but it is hardly plausible that they refuse to agree about anything in order to preserve their livelyhood. We might ask, then, why is it we are still in such a quandry over the simplest philosophical problems? In search of an answer to this question, let us start by comparing philosophy to another science.

A modern physicist does not have to read the works of Newton or Archimedes to be respected as a good physicist. If he were to read Newton or Archimedes, he would do so out of historical interest. Much of what Newton and Archimedes wrote has since been discredited. A modern philosopher, on the other hand, must be as familiar with the work of the Greeks as he is with the work of the twentieth century philosophers. The philosophical ideas of the Greeks have not been discredited by any advances in the field of philosophy: one is not considered a fool for supporting, for example, Aristotle's Nicomachean Ethics. A physicist, however, would certainly be regarded as a fool for believing in Newton's Corpuscular Theory of Light. Many argue that this difference between philosophy and physics exists because it is impossible to solve any philosophical problem in such a way as to convince everyone. Of course, this point cannot be argued in such a way as to convince everyone, for if it could, then one would be able to do something one had just proved was impossible — namely, to convince everyone of some philosophical point.

With this book, I hope to show that the difficulty we have so far faced in supplying answers to philosophical questions arises not so much out of the nature of those questions as out of our failure to be disciplined and clear in our philosophical contemplations. Notice, for example, that there is a great difference between the writings of physicists and the writings of philosophers. When we study physics, our mental efforts are directed towards training our minds to apply the physical theories. We are never in any doubt as to what those theories are: they are expressed clearly and unambiguously in our textbooks. When we study philosophy, our mental efforts are directed towards interpreting the words of philosophers. While physics students are tested on their ability to apply the theories descibed in their textbooks, philosophy students are asked to write essays to show how good they are at interpreting philosophical texts.

I propose that severe problems arise in philosophy as a result of its distraction with interpretation. Consider an analogous dilemma, which arises when we deprive accountants of their numbers. Numbers are tools we have invented to allow us to communicate ideas about quantities. They are abstract. The number 3, for example, cannot be found in this world. There is a mathematical symbol for it, there is a spoken word for it, and we each have an idea of it; but we do not have the number itself.

Without numbers, accounting would be a precarious trade; entangled disagreements would arise about how much was owed. Of course, people would reach agreement occasionally, and these successes would encourage everyone else to keep trying. Those who tried hardest to solve the problems of how much was owed would be called 'accountants'. But, without numbers, the accountants would have difficulty communicating their ideas. Consequently, they would spend a lot of their energy trying to make themselves clear to one another. There would be little hope of dismissing any particular accounting theory, since there would be no numbers with which to express a clear enough argument against it. However, everyone would recognise that the problems of accounting needed to be solved, so they would pay the accountants to keep thinking. Accountants without numbers would have a lot in common with our philosophers.

I suggest, then, that philosophy is missing some fundamental tools of trade. Consequently, philosophers do not think clearly enough to solve complex philosophical problems. Physicists, on the other hand, are equipped with all the tools of mathematics, and so are able to find simple laws to account for complex physical phenomena.

In this book I shall describe two philosophical tools. I call them 'frames' and 'symbolisms'. Frames and symbolisms are to reasoning and language what numbers are to quantities. They can be to philosophers what numbers are to accountants. They are merely tools; they tell us nothing about the universe; they make no demands upon us to adopt any particular way of thinking. And yet they may be used to describe any use of reason or language.

I might have chosen to define frames and symbolisms with standard English prose, or to communicate their properties by providing many examples of their application. These definitions might have been more palatable to my readers, but instead I have chosen to define frames and symbolisms using a branch of mathematics called 'set theory'. Set theory is the basis of modern mathematics. I shall try to justify my choice of set theory by discussing the problem of defining numbers, for this is a problem to which set theory provides an elegant solution. You need not be concerned if you do not know any set theory. You can learn all you need from Chapter 2.

We learn about numbers when we are children. We hear people talking about them, we are taught to do sums and to memorise our multiplication tables. These things teach us the properties of numbers. We are not told about the numbers themselves. Consequently, while we all agree upon the mathematical properties of numbers, we may have very different ideas about their other properties. I think 3 is green. My sister thinks it is red. Others don't think it has any colour at all. But we all agree that $3 \times 3 = 9$ (which is brown). The mathematical properties of numbers are important, while their other properties are not.

Mathematicians need a more succinct definition of the properties of numbers than the one given to children. Mathematicians use many different 'number systems'. The 'integers', the 'rational numbers', and the 'real numbers' are three examples of number systems. Even the slightest discrepancies between their understandings of the properties of a number system would lead mathematicians into intolerable confusion. Therefore, they have devised ways of condensing all the properties of a number system into a short list of statements. These statements are called the 'postulates' of the number system. The properties of a number system are the logical consequences of its postulates. For example, suppose we have:

Postulate 1.1 For every number, y, we have $1 \times y = y$.

If we want to know what is 1×3, we can use logic to let y represent 3 in Postulate **??**, and so conclude that $1 \times 3 = 3$. If we want to know what is 1, or 1×4, then use Postulate **??** in the same way. We would find that $1 \times 2 = 2$ and $1 \times 4 = 4$. If some day we should want to know what is 1×999, then we shall be able to use Postulate **??** to find out. Postulate **??** saves us from writing down an infinity of statemens '$1 \times 1 = 1$', '$1 \times 2 = 2$', '$1 \times 3 = 3$', etc. The rules of logica allow us to arrive at all these statements from a single postulate.

If we wanted to conclude also that $3 \times 1 = 3$ (the order has changed), then we would need another postulate:

Postulate 1.2 For every two numbers, x and y, we have $x \times y = y \times x$.

A list of postulates will not work if it leads to a logical contradiction. For example, Postulates **??** and **??** do not work in combination with the postulate 'For every number, y, we have $y \times 1 = 1$'.[1]

Mathematicians have devised working lists of postulates for all of their number systems. A property deduced from a working list of postulates is called a 'theorem'. From Postulates **??** and **??** alone, we could prove the following theorem:

Theorem 1.3 For every number, y, we have $y \times 1 = y$.

Towards the end of the nineteenth century it was discovered that complicated abstract objects, such as numbers, could be defined in terms of much simpler abstract objects called 'sets'. Numbers are defined by constructing special sets which behave like numbers and then saying that the numbers are one and the same as the sets which have their properties. For example, in one system, the 'empty set' is found to have the properties of the number zero, so we just say that the number zero *is* the empty set — or, 'zero' and 'empty set' mean the same thing. This is not a conception of numbers any child is likely to invent for himself, but that does not matter, since a mathematician still agrees that $3 \times 3 = 9$. For the mathematician, the properties of numbers are layed down in terms of the much simpler properties of sets. The properties of sets are, of course, defined by a list of postulates, and set theory is the study of these postulates. Mathematicians have been able to construct all mathematical objects out of sets. Numbers are just particular sets with particular names — although it should be noted that these sets can be complicated.

The philosophical tools I would like to describe to you can be defined elegantly in terms of sets. By investing in a chapter on set theory, I have been able to make this a short book. If you are unused to mathematical thinking, then the next chapter may initially be difficult for you to understand. I hope you will not be discouraged, for once you have grasped the fundamental properties of sets, and grown used to using mathematical logic, the rest of the book will make for much simpler reading. Moreover, you will have the pleasure of knowing that you understand the founding principles of mathematics. It does not matter if you distrust mathematical logic as a way of thinking. I shall make no effort to suggest that mathematical reasoning is superior to any other type.

[1] To see why, let $y = 3$. Then Postulate **??** states that $1 \times 3 = 3$, and Postulate **??** states that $1 \times 3 = 3 \times 1$. This means $3 \times 1 = 3$, but the newly suggested postulate says $3 \times 1 = 1$, which is logically contradictory to $3 \times 1 = 3$.

Chapter 2

Set Theory

Set Theory is the study of the postulates and axioms of sets. The set postulates define the properties of sets. The set axioms determine which sets are available to us. In this chapter I shall present the axioms of what is called the 'Zermelo-Fraenkel Set Theory'.

2.1 Sets

Let us begin with an informal description of sets. We introduce the notion of a 'property'. An example of a property is 'being an apple' or 'being a fraction'. For the sake of brevity we adopt a policy of representing properties with letters. If we choose the letter P to represent the property 'being an apple', then saying something has property P is the same as saying it is an apple.

The 'set of all things with property P' is a fantasy box containing everything with property P and nothing else. The fantasy box has no physical existence. It is abstract. It does not exist in the mind or on paper. In our minds we imagine it. On paper we represent it.

Sets have the property of being outwardly identical regardless of the things they contain. The only way of telling two sets apart is by looking inside them and examining their contents. The things in a set are called its 'members'. If two sets have the same members, then they are identical. We say they are 'equal' sets. We find ourselves referring to *the* set of all things with property P rather than *a* set of all things with property P.

The members of a set could be things with physical existence, or abstract things, or fictional things, or a mixture of all three. There is a set with no members. It is called the 'empty set'. The empty set is important in mathematics. There is a set whose only member is the empty set, and a set whose members are the empty set and the set containing the empty set. We can go on forever in this way, generating sets which are outwardly identical, but whose contents are increasingly more complicated. This is how mathematicians construct sets whith the properties of numbers. They start with the empty set and construct sets whose contents are structured in such a way as to give them the properties of numbers.

Mathematics does not rely for its validity upon any assumption about the physical universe. Mathematics is purely abstract. No matter what we believe about the universe, mathematics will never offend our beliefs. It has no purpose other than to provide us with abstract tools. We use these tools in whatever way we see fit.

This chapter is devoted to the mathematical properties of sets, and the manner in which sets may be used to represent complex abstract objects. The properties of sets are defined by two postulates. The 'logical consequences' of these postulates are the properties of sets. I shall devote the next few sections of this chapter to a brief description of logic.

2.2 Logic

We are going to define a 'logical system'. The first step in defining a logical system is to choose a language in which to phrase statements. Once we have chosen a language, we select a set of statements in that language, and call it our 'basis set'. The rules of logic tell us which statements in our language are the logical consequences of our basis set. The logical consequences of a basis set are also called its 'theorems'.

For the language of our logical system, we shall use a mixture of English and Mathematical statements. I refer to this mixture as 'Mathematical English'. A statement in Mathematical English is any sequence of one or more grammatically and syntactically correct Mathematical English sentences. We shall not use all the statements of Mathematical English. Our language will be a restricted version of Mathematical English, which I shall refer to as 'Restricted Mathematical English'. Restricted Mathematical English includes all statements of Mathematical English except those whose meaning is akin to, *'This statement is not a theorem.'* We exclude all statements of this type to avoid logical paradoxes which would otherwise detract from the value of our logical system. We still allow statements like, *'Let A be the statement "A is not a theorem".'* From this statement we might be tempted to argue that if A is a theorem then it is not a theorem, and if it is not a theorem then it is a theorem. It turns out that such an argument relies upon an abuse of mathematical symbolism. If we were to use the symbol 'B' instead of 'A', while still referring to the statement, *'A is not a theorem,'* then the argument would fall apart.

The members of a basis set are called 'basis statements'. Basis statements are separated into 'axioms' and 'definitions'. An axiom is used to represent an idea we assume is a fact. A definition is used to define terminology. A 'postulate' is a fragment of a definition.

The most fundamental rule of logic is the 'basis rule':

Basis Rule: Every basis statement is a theorem.

To abbreviate my description of the remaining rules of logic, I shall use the symbols 'A', 'B', and 'C' to represent Restricted Mathematical English statements. The symbol 'A' might represent, *'I see a bright orange thing in the sky,'* while 'B' might represent, *'The sun is up,'* and 'C' might represent, *'It is daytime.'* The expression, *'If A then B,'* would represent, *'If I see a bright orange thing in the sky then the sun is up.'* The expression 'AB' would represent, *'I see a bright orange thing in the sky. The sun is up.'*

Now we can define the rule of 'conjunction' using the symbols 'A' and 'B':

Conjunction: If 'A' and 'B' are theorems, then 'AB' is also a theorem.

If 'A', 'B', and 'C' are theorems, then so are 'AB', 'BC', and 'AC'. Applying the rule of conjunction to 'AB' and 'C' we find that 'ABC' is another theorem.

Mathematical statements often include the use of shorthand. When mathematical shorthand is used in a statement, there will be clauses like, *'Let x be such-and-such,'* or, *'If y is such-and-such.'* Such clauses are called 'assignment clauses'. Statements which use mathematical shorthand are called 'indirect statements', while statements which make no use of mathematical shorthand are called 'direct statements'. The next logical step applies only to direct statements:

Extraction: If 'A' and 'B' are two direct statements such that 'AB' is a theorem, then 'A' and 'B' are also theorems.

If the statement, *'The sun is up. It is not raining,'* is a theorem, then so is the statement, *'It is not raining.'* We are not allowed to apply extraction to indirect statements. To see why this is so, suppose the following indirect statement were a theorem: *'Let x be the diameter of the moon, and let y be the* *diameter of the earth. We have x less than y.'* If we could apply extraction to this statement, then, *'We have x less than y,'* would be a theorem. But what do x and y represent in this statement? The symbols x and y had meaning in the indirect statement because they had been defined by an assignment clause, but on their own, x and y are meaningless. If we were to apply extraction to indirect statements, we would generate meaningless theorems.

Transitive Implication: If 'A implies B' and 'B implies C' are theorems, then 'A implies C' is also a theorem.

The statement 'A implies B' is equivalent to 'If A then B'. The statement 'A implies B' means the same as, *'If A is a theorem then B is also a theorem.'*

Every statement in English has at least one 'negation' statement. The statement, *'It is not daytime,'* is a negation of, *'It is daytime.'* Conversly, *'It is daytime,'* is a negation of, *'It is not daytime.'* Turning to a more complicated example, the statement, *'Let x and y be the diameters of the moon and earth respectively. We have x greater than y,'* has the negation, *'Either x does not represent the diameter of the moon, or y does not represent the diameter of the earth, or x is less than or equal to y.'* [1]

We use the symbol '$\overline{A}$' [2] to represent a negation of 'A'. If '$\overline{A}$' is a negation of 'A', then 'A' is likewise a negation of '$\overline{A}$'. Negation statements are used in the following logical step:

Contrapositive Statement: If 'A implies B' is a theorem, then any statement of the form '$\overline{B}$ implies $\overline{A}$' is also a theorem.

A statement of the form '$\overline{B}$ implies $\overline{A}$' is called a 'contrapositive statement' of 'A implies B'. The statement, *'If it is not daytime then I do not see a bright orange thing in the sky,'* is a contrapositive statement of, *'If I see a bright orange thing in the sky then it is daytime.'*

In formal logic, we have syntactic rules which tell us which statements are 'logically equivalent' to one another. Then we can use the following rule of logic:

Logical Equivalency: If 'A' is a theroem, and 'B' is logically equivalent to 'A', then 'B' is a theorem.

In English, there are a vast number of synonymous words, phrases, and arrangements of phrases. I do not have space in this book to define the rules of logical equivalency in our logical system. To do so, I would have to start with the logical equivalency of 'yesterday' and 'the day before today', and then move on through all the synonyms of English, before I even began to define the equivalent arrangements of phrases which are supported by Restricted Mathematical Enlgish. Therefore I shall assume that we share an understanding of the logical equivalency of Restricted Mathematical English statements. However, I shall provide a few examples of the logical equivalencey of indirect statements.

The statement, *'For all x we have $x = x$,'* is logically equivalent to the following statements:
(1) 'For all y we have $y = y$.'
(2) 'For any z we have $z = z$.'
(3) 'If θ exists then $\theta = \theta$.'
(4) 'Everything is equal to itself.'
The statement, *'Let A and B be sets. We say A is a subset of B if and only if every member of A is a member of B,'* is logically equivalent to the following statements:
(1) 'We say a set, x, is a subset of another set, y, if and only if every member of x is a member of y.'

[1] Here the word 'or' is used in the logical sense: 'A or B' means, 'A alone, B alone, or A and B together.'
[2] Read 'not A'.

(2) 'Let A and B be sets. Every member of A is a member of B if and only if A is a subset of B.'
(3) 'A first set is a subset of a second if and only if every member of the first set is also a member of the second set.'
The statement, *'If A is a set and x is chosen arbitrarily, then x cannot both be a member of A and not a member of A,'* is logically equivalent to the following statements:
(1) 'For any set, T, and any thing, t, we cannot have both t a member of T and t not a member of T.'
(2) 'If Π is a set, then there is no π which is both a member of Π and not a member of Π.'
(3) 'Given any set and any thing, the thing cannot both be a member of the set and not a member of the set.'
For every statement in Restricted Mathematical English, there is a logically equivalent direct statement. These logically equivalent direct statements may be cumbersome, but they exist.

We say a basis set is 'logically contradictory' whenever it has a logical consequence of the form '$A\overline{A}$'. For example, suppose we find we have the following two theorems: *'It is raining,'* and, *'It is not raining.'* The rule of conjunction tells us, *'It is raining. It is not raining,'* is a theorem, so our basis set is logically contradictory. We assume that a basis set containing no basis statements has no logical consequences at all, for there are no basis statements to which we can apply the rules of logic. Therefore an empty basis set cannot be logically contradictory. A basis set can be logically contradictory only if it contains at least one statement. This means that when we find a basis set to be logically contradictory, we assume it is because of some conflict between the basis statements. If a basis set containing only one basis statement is found to be logically contradictory, then we say the basis statement is a 'logical paradox'. An example of a logical paradox is the statement, *'It is raining. It is not raining.'*

Logic is an impersonal system of thought. The rules do not allow the user to make leaps of reasoning unless those leaps are justified by the rules of logic. From the statement, *'If dragons once roamed the world, then we would have legends of dragons,'* logic does not permit us to conclude that, *'If we have legends of dragons, then dragons once roamed the world.'* To do so would be to 'argue the converse'. Arguing the converse is when we use a statement 'A implies B' to conclude that 'B implies A'. This is not allowed in logic.

Another type of reasoning forbidden by logic is 'argument by analogy'. Argument by analogy is where obvious similarities between things are taken as proof of further similarities. Here is an example of argument by analogy:

'Nature is an intricate and smoothly running system, just like a watch. All watches are designed and created by someone, so nature must have been designed and created by someone. Furthermore, nature is infinitely more intricate than a watch, so its creator must be infinitely more intelligent than man.'

Many thinkers find logic too restrictive for their purposes. They turn to other systems of thought. In this book I use logic as a way of abbreviating the description of abstract objects. Logic serves this purpose well — even if it is inadequate in other respects. You need lose no faith in my descriptions just because you do not consider logic to be an adequate tool for thinking about philosophical problems.

Moreover, the systems of thought used by those who distrust logic tend to differ from logic only in that these systems allow steps of reasoning in *addition* to those provided by logic. Such systems of thought might allow one to use argument by analogy, or argument of the converse. When we choose an alternative system of reasoning, we tend to make it less stringent than logic rather than more stringent. Consequences of a basis set in an alternative system of reasoning are likely to include all the logical consequences of that basis set as well as many other consequences. In this book, I use logic frequently to make sense of the way I have defined words and arranged philosophical tools. I trust that these justifications, being logical, will be agreeable to my reader — not because I trust you are an advocate of logic, but because I suspect that your preferred system of reasoning will at least include the rules of logic.

2.3 Logical Existence

Any mention of a thing in any theorem implies that the thing 'exists' in our logical system — unless the thing is mentioned only in conditional clauses, or clauses which declare that the thing does not exist. For example, we can say, *'If the moon exists...,'* or, *'If x is the moon...,'* or, *'For all x such that x is the moon we have...,'* and not imply the existence of the moon in our logical system. We also can say, *'There is no moon,'* and not imply the existence of the moon. However, any of the following clauses, if found in theorems, would imply the existence of the moon in our logical system: *'Let x be the moon,'* or, *'Let x be the diameter of the moon,'* or, *'The diameter of the moon is less than the diameter of the earth.'* Whenever a thing is found to exist in our logical system, We can include among our theorems a statement declaring the thing's existence.

Declarations of Existence: If a thing exists in a logical system, then there is a theorem declaring its existence.

If the statement, *'The largest living brontasaurus lives in Montana,'* is a theorem, then we also have the following two theorems: *'There exists a largest living brontasaurus,'* and, *'There exists a Montana.'*

In the expression, *'For every x we have...,'* the 'every' refers to everything which exists in the logical system. When we say, *'For every set, A, and every property, P, we have...,'* we mean, *'If A is any set which exists, and P is any property which exists, then we have...'* The above statement does not apply to sets and properties which do not exist in our logical system.

2.4 Theorems and Proofs

A 'proof' of a theorem is a demonstration of how the rules of logic may be used to arrive at the theorem from the basis statements. In theorem proofs we make liberal use of the words 'so', 'therefore', 'thus', and 'implies'. These words all mean the same thing. They are equivalent to saying: 'By taking finitely many logical steps we can reach the following statement...'

It is traditional to use mathematical shorthand in theorem proofs. This makes them easier to understand. The mathematical shorthand we use in theorem proofs is the same as the shorthand we use for the abbreviation of theorems, axioms, and definitions. However, we must be careful when we use shorthand in our proofs. A proof is supposed to be a demonstration of how the rules of logic may be used to move from one theorem to another until we arrive at the theorem we have set out to prove. In a valid proof, symbols represents the application of logic to theorems. The shorthand statements of a proof are not themselves theorems. They are merely representations of theorems, and these representations have meaning only in the light of the declarations of shorthand that are found in the proof. Returning to the example of page **??**, consider the following attempt at a proof:

'Let A be the statement, "A is not a theorem". If A is a theorem, this means "A is not a theorem" is a theorem, which means A is not a theorem. So our basis set is logically contradictory if A is a theorem. Thus if our basis set is not logically contradictory, then A cannot be a theorem. If A is not a theorem, then the statement, "A is a not a theorem" is a theorem, so A is a theorem. So our basis set is logically contradictory in A is not a theorem. Thus if our basis set is not logically contradictory, then A must be a theorem. Combining these two results, we see that if our basis set is not logically contradictory, then A is a theorem and A is not a theorem, so we find our basis set is logically contradictory. Therefore, all basis sets must be logically contradictory.'

This proof may at first sound credible, but it is vacuous. The proof does not work if we begin it with the statement, *'Let B be the statement "A is not a theorem",'* instead of, *'Let A be the statement "A is not a theorem".'* The use of the symbol 'A' leads us astray unless we recognise that we are not entitled to say that the 'A' in, *'A is not a theorem,'* means the same as the 'A' we use for shorthand. The two occurances of the symbol can mean the same thing only when we have the theorem, *'There exists a statement, A, which is the statement "A is not a theorem".'* This theorem declares the existence of a thing with two permanent names. One of these names is 'A', and the other one is 'the statement "A is not a theorem".' Attaching these names to the same thing in our logical system makes our basis set logically contradictory.

2.5 The Beginnings of a Basis Set

In this section, we begin the construction of a basis set. Ultimately, this formal basis set will define a logical system in which there exist numbers, mappings, frames and symbolisms. At the beginning of our construction, our formal basis set is empty. In this book there are axioms, definitions, and postulates presented formally in numbered paragraphs. These are additions to our formal basis set. There are also numbered paragraphs which present 'theorems'. These theorems are logical consequences of our formal basis set. Each numbered theorem is followed by a formal paragraph presenting a proof of the theorem.

The postulates presented in Chapter 1 are only example postulates. They do not count as part of our formal basis set. At the moment, our formal basis set is empty. The first statement we add to it is Axiom **??**. Before we declare this axiom, let us for brevity's sake define a symbol to represent our formal basis set in the text. The symbol 'Δ' represents the set of all numbered definitions, axioms and postulates introduced from the beginning of Chapter 2 up to the point in the book where the symbol is used. Therefore, Δ now represents the empty set. When used at the beginning of Chapter 3, Δ will represent the set of all numbered definitions, axioms, and postulates (but not theorems) introduced in Chapter 2.

Each formal paragraph in the book has a unique number. The first formal paragraph in Chapter 2 is numbered 2.1, the second is numbered 2.2, and so on, irrespective of the type of the paragraph. If we have Theorem 2.30 this does not mean it is the thirtieth theorem in Chapter 2, but that it is the thirtieth numbered paragraph. It could be the first theorem. The preceding twenty nine paragraphs could be postulates, definitions and axioms.

It may seem beyond question that all things are equal to themselves, but the statement of this unquestionable fact is our first axiom: the Axiom of Equality.

Axiom 2.1 (Equality) For every x we have $x = x$.

When we say 'x equals y,' we do not mean the letters x and y are the same, but rather that the things they represent are the same. All of the following statements are logically equivalent:
(1) The symbols x and y both represent the same thing.
(2) x and y are equal.
(3) $x = y$.
(4) x is y.

Axiom **??** states that everything which exists in our logical system has the property of being equal to itself. From it we can prove the following theorem:

Theorem 2.2 There exists no x such that $x \neq x$.

The phrase, '*there exists no x such that...*,' means, '*there does not exist any thing, x, such that...*' The phrase '$x \neq x$' means, '*x is not equal to x.*' The backslash through a symbol is standard mathematical notation for the negation of the symbol. Here is one way of proving Theorem **??**.

Proof of Theorem 2.2 Axiom **??** states that for every x we have $x = x$, so the rule of logical equivalency states that if x exists then $x = x$, so the rule of contrapositive statement states that if $x \neq x$ then x does not exist, so the rule of logical equivalency states that there exists no x such that $x \neq x$. □

The proof argues that two applications of the rule of equivalency, and one application of the rule of contrapositive statement are enough to arrive at Theorem **??** from Axiom **??**.

2.6 Notation for Sets

A property is 'paradoxical' if and only if for some x we can prove both that x has the property and that x does not have the property.[3] A property is 'well-behaved' if and only if it is not paradoxical. A property is 'definite' if and only if for every x we can prove either that x has property P or that x does not have property P, but not both. We would like to be able to use well-behaved properties as instruments in our logical proofs. The following axiom declares that all well-behaved properties exist in our logical system.

Axiom 2.3 (Existence of Properties) A property, P, exists if and only if there exists no x such that x both has property P and does not have property P.

Axioms **??** and Axiom **??** together imply the existence of the following two properties in our logical system:
(1) P is the property such that x has property P if and only if $x = x$.
(2) Q is the property such that x has property Q if and only if $x \neq x$.
Axiom **??** implies everything has property P, while Theorem **??** implies nothing has property Q. Thus both P and Q are definite, and consequenctly well-behaved. Thus Axiom **??** states that both P and Q exist.

Here is our first formal definition. It defines some notation for use in our discussion of sets.

Definition 2.4 (Sets) We say $A = \{x \mid x \text{ has property } P\}$ if and only if A is the set of all things with property P.

Curly brackets are used to enclose the definition of a set. We can imagine them representing the walls of the fantasy box. The expressions between the curly brackets tell us what is in the box. The expression, '$A = \{x \mid x \text{ has property } P\}$,' reads as follows: '$A$ is equal to the set of all things, x, such that x has property P.' The '$|$' means 'such that'. Thus $\{x \mid 0 < x < 1 \text{ and } x \text{ is a fraction }\}$ is the set of all fractions greater than zero but less than one.[4] We do not have to use x as the symbol representing things in the set. Any symbol will do.

Definition 2.5 (Membership of Sets) If A is a set, then we say *x is a member of A*, denoted $x \in A$, if and only if A is the set of all things with property P and x has property P.

[3] As an example of a paradoxical property, consider a property named P which is the property of not having property P.

[4] The symbol '$<$' means 'less than'. Also, '$\leq$' means 'less than or equal to', '$>$' means 'greater than', and '$\geq$' means 'greater than or equal to'.

The statement 'x is a member of A' could mean 'Jason is a Mason'. To say 'x is a member of A' does not imply A is a set. Nor does '$x \in A$'. Definition **??** has nothing to say about the membership of anything other than sets. We shall use the symbol '$\notin$' to mean 'not a member of'.

Here is another way of defining simple sets.

Definition 2.6 (More Symbols For Sets) If A is a set, then we say $A = \{x, y\}$ if and only if $A = \{z \mid z = x \text{ or } z = y\}$.

Thus $\{x \mid x = 1 \text{ or } x = 2\} = \{1, 2\}$. We could represent the empty set as $\{\}$.

Definition 2.7 (Membership Property of a Set) If A is a set, then the *membership property* of A is the property of being a member of A.

The membership property of $\{x \mid x \neq x\}$ is simply 'being a member of $\{x \mid x \neq x\}$'. This property is different from the property 'being a living brontasaurus'. It is only because our logical system does not contain a brontasaurus that the sets of things with these properties are the same set (the empty set). If we were to add to our basis set a statement declaring the existence of a brontasaurus, then the sets of things with these two properties would no longer be the same. One of the sets would contain a brontasaurus, and the other would still be empty.

2.7 The Set Postulates

Sets are distinguished by their members. Two sets are different only if there is a member of one set which is not a member of the other set. This is the first of our set postulates.

Postulate 2.8 (Equality for Sets) If A and B are sets, then we say $A = B$ if and only if there is no member of A which is not a member of B and there is no member of B which is not a member of A.

Now we have defined formally what it is for two sets to be equal, we need only equip ourselves with the axiom *'There exists no living brontasaurus,'* and we can use Postulate **??** and Axiom **??** to prove formally that the sets $\{x \mid x \neq x\}$ and $\{x \mid x \text{ is a living brontasaurus }\}$ are equal.

If A is a set, then we would like to be sure that there is no x for which we can prove both $x \in A$ and $x \notin A$. In other words, we would like the membership property of all sets to be well-behaved. Because of Axiom **??**, a property is well-behaved if and only if it exists.

Postulate 2.9 (Well-Behaved Membership) If A is a set, then the membership property of A exists.

We have defined the properties of sets. Now we shall add the set axioms to Δ. The set axioms tell us which sets exist in our logical system. The first five of the set axioms are called the 'axioms of the basic set operations'. These axioms do not themselves declare the existence of a set. Instead they are similar to the logical steps in their application. The logical steps do not declare the existence of any particular theorem. They merely tell us which theorems will accompany existing theorems. Similarly, the axioms of the basic set operations do not declare the existence of any particular set. They merely tell us what sets will acompany existing sets.

2.8 The Axioms of the Basic Set Operations

We create sets which act as representations of complex abstract objects by constructing them gradually out of simple sets. This is done with a sequence of 'set operations'. The five fundamental set operations are called the 'basic set operations'. All set operations may be expressed in terms of the basic set operations. The first basic set operation we encounter is called 'comprehension'. Comprehension is where we take a set, A, and a property, P, and define a set $C = \{x \mid x \in A \text{ and } x \text{ has property } P\}$. The set C is the set of all members of A which have property P.

If A is a set, then any set which can be created out of it using a sequence of basic set operations is called a 'relative of A'. The basic set operations are such that if A is a relative of B, then B is also a relative of A. That is, if B can be created out of A using the basic set operations, then A can likewise be created out of B.

We intend to declare the existence of a simple set, and then show that its relatives have properties useful to us. We would like to be sure that all the relatives of our simple set exist in our logical system, so that we can prove all manner of theorems about them. To this end, we have the five 'axioms of the basic set operations'. There is one such axiom for each of the basic set operations. For example, if A is a set, and B is another set related to A by comprehension, then the Axiom of Schema Comprehension states that B exists whenever A exists. Any set related to B by comprehension will also exist, for we may apply the Axiom of Schema Comprehension to B as well. There is one set axiom for each basic set operation, so all the immediate relatives of A exist. All of their immediate relatives exist too, as do all immediate relatives of those immediate relatives — and so on, for as long as we like, so that all relatives of A exist.

Axiom 2.10 (Schema Comprehension) If A is a set and P is a property, then there exists a set $\{x \mid x \in A \text{ and } x \text{ has property } P\}$.

Mathematicians regard Axiom **??** as an infinity of axioms, each corresponding to a different existing property. This is why they have called it 'schema comprehension' rather than just 'comprehension'.

The set operation called 'intersection' is not one of our basic set operations, but it may be defined in terms of comprehension.

Definition 2.11 (Intersection of Sets) If A and B are sets, then C is the *intersection of A and B*, denoted $A \cap B$, if and only if $C = \{x \mid x \in A \text{ and } x \in B\}$.

Thus $\{1,2\} \cap \{2,3\} = \{2\}$. In mathematics, the operation of intersection is used frequently — as is the operation of 'union' (see Definition **??**). If you have had any experience with the mathematics of sets, then you will be familiar with the operations of intersection and union. You may have expected them to be included among the basic set operations, but they are not.

Theorem 2.12 (Existence of Intersections) If A and B are sets, then $A \cap B$ exists.

Theorem **??** does not say any set exists. It merely says that if two sets were to exist, then the set which is their intersection would also exist. To prove Theorem **??**, we show that whenever there exist two sets A and B, it is a theorem that $A \cap B$ exists. This is the same as saying, 'If A and B are sets, then $A \cap B$ exists.'

Proof of Theorem 2.12 Suppose there exist sets A and B. Then Postulate **??** states that the membership property of B exists. Call it P. A is a set, so Axiom **??** states that $\{x \mid x \in A \text{ and } x \text{ has property } P\}$ $= A \cap B$ exists. □

Now we define the word 'subset' to indicate a particular relationship between two sets.

Definition 2.13 (Subsets) If B is a set, then we say A is a *subset* of B, denoted $A \subset B$, if and only if, A is a set such that there is no member of A which is not a member of B.

The next theorem is an example of an 'if and only if' theorem.

Theorem 2.14 If A and B are sets, then $A = B$ if and only if both $A \subset B$ and $B \subset A$.

Theorem **??** says not only that $A = B$ implies both $A \subset B$ and $B \subset A$, but also that $A \subset B$ and $B \subset A$ together imply $A = B$. The proofs of such theorems are split into two parts. First we prove the implication in one direction (the 'if'), then we prove it in the other direction (the 'only if').

Proof of Theorem 2.14 Suppose there are two sets A and B such that $A \subset B$ and $B \subset A$. Then Definition **??** states that there is no member of A which is not a member of B, and there is no member of B which is not a member of A. Thus Postulate **??** implies $A = B$. Now, to prove the converse, suppose there exist two sets A and B such that $A = B$. Then Postulate **??** implies there is no member of A which is not a member of B, so by Definition **??** we have $A \subset B$. Postulate **??** also implies there is no member of B which is not a member of A, so by Definition **??** we have $B \subset A$. □

If A is a set, then the 'power set' of A is the set of all subsets of A. Creating the power set of an existing set is the second of our five basic set operations.

Definition 2.15 (Power Sets) If A is a set, then we say P is the *power set of* A, denoted $\wp(A)$, if and only if, $P = \{x \mid x \subset A\}$.

The power set of $\{1, 2\}$ contains $\{1, 2\}$, $\{1\}$, and $\{2\}$. It also contains the empty set, since the empty set is a subset of every other set (see Theorem **??**). Thus we have:

$$\wp(\{1, 2\}) = \{\ \{\}\ , \{1\}, \{2\}, \{1, 2\}\}.$$

Here is the set axiom saying the power set of any existing set also exists:

Axiom 2.16 (Power Set) If A is a set then the power set of A exists.

If $\{1, 2\}$ existed in our logical system, then Axiom **??** implies $\wp(\{1, 2\})$ would exist as well.

The third basic set operation is called 'pairing'. Pairing is where we take two sets, A and B, and make $C = \{A, B\} = \{x \mid x = A \text{ or } x = B\}$.

Axiom 2.17 (Pairing) If A and B are sets then there exists a set $\{A, B\}$.

The fourth basic set operation is called 'internal union'. Internal union is illustrated by the following example: Let $A = \{B, C\}$, where B and C are sets. We create a new set $D = \{x \mid x \in Y \text{ for some set } Y \in A\} = \{x \mid x \in B \text{ or } x \in C\}$. Thus D is the set of things which are members of some set which is itself a member of A. We say D is the 'internal union' of A.

Axiom 2.18 (Internal Union) If A is a set then there exists a set $\{x \mid x \in Y \text{ for some set } Y \in A\}$.

While the operation of pairing takes sets and makes a set of sets, the operation of internal union does the opposite. It is possible to show that, given any basic set operation, we can construct its opposite out of the basic set operations. Thus we are certain that whenever A is related to B, B is also related to A, for we are sure that whenever we can create B out of A, we can also go backwards along each step of the creation, and so return to A.

Here is a commonly used set operation which, like intersection, may be expressed in terms of our basic set operations.

Definition 2.19 (Unions of Sets) If A and B are sets, we say C is the *union of* A *and* B, denoted $A \cup B$, if and only if $C = \{x \mid x \in A \text{ or } x \in B\}$.

Thus $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$. In the next theorem Axioms **??** and **??** are used to prove that the union of two sets exists.

Theorem 2.20 (Existence of Unions) If A and B are sets then $A \cup B$ exists.

Proof of Theorem 2.20 Suppose there exist sets A and B. Axiom **??** implies $C = \{x \mid x = A \text{ or } x = B\}$ exists. C is a set, so Axiom **??** implies $\{x \mid x \in Y \text{ for some set } Y \in C\} = \{x \mid x \in A \text{ or } x \in B\} = A \cup B$ exists. □

Given any two different numbers, we know one of them will be less than the other. Mathematicians say numbers are 'ordered'. Postulate **??** implies $\{x, y\} = \{y, x\}$, so $\{x, y\}$ does not represent any ordering of x and y. Here is a way to represent order with sets:

Definition 2.21 (Ordered Pairs) For any x and y, the set $\{\ \{x\},\ \{x, y\}\ \}$, denoted (x, y), is an *ordered pair.*

Given $(x, y) = \{\{x\},\ \{x, y\}\}$ with $x \neq y$, we can tell x is placed before y in the ordering, because there is a member of (x, y) which contains x but not y. As an example of the use of ordered pairs, let x represent the latitude of a ship's position, and y the longitude. We let $p = (x, y)$ represent the position of the ship. We could not have used $p = \{x, y\}$ because we would not know which of x and y was latitude, and which was longitude. The set $\{x, y\}$ is not ordered. Let us go further with our example, and define P to be the set of all positions on the surface of the earth. Thus $P = \{(x, y) \mid x \text{ is a latitude and } y \text{ is a longitude}\}$. Note the slightly different notation I have used to define P. It means the same as $\{p \mid p = (x, y) \text{ where } x \text{ is a latitude and } y \text{ is a longitude}\}$.

Theorem 2.22 (Existence of Ordered Pairs) If $\{x\}$ and $\{y\}$ are sets, then (x, y) exists.

Proof of Theorem 2.22 Suppose x and y are such that $\{x\}$ and $\{y\}$ exist. Theorem **??** implies $\{x,\ y\}$ is exists. $\{x,\ y\}$ and $\{x\}$ are sets, so Axiom **??** implies $\{\{x\},\ \{x,\ y\}\} = (x, y)$ exists. □

Ordering is readily extended to three or more things. Here is one way of ordering three things. It makes use of ordered pairs.

Definition 2.23 (Ordered Triples) For any x, y, and z, the set $((x, y),\ z)$, denoted (x, y, z), is an *ordered triple.*

The ordered triple (x, y, z) is a complicated set:

$$(x, y, z) = \{\ \{\{\{x\}, \{x, y\}\}\},\ \{\{\{x\}, \{x, y\}\}, z\}\ \}.$$

However, it is a simple matter to prove (x, y, z) exists whenever $\{x\}$, $\{y\}$, and $\{z\}$ exist.

The fifth and last of our basic set operations is called 'replacement'. This is where one replaces every member of a set with something else which exists in the logical system. We express this replacement mathematically with the help of ordered pairs. Here is the axiom saying that any set generated by replacement exists in our logical system.

Axiom 2.24 (Schema Replacement) If A is a set and P is a property such that for every $x \in A$ there is a unique y for which (x, y) has property P, then there exists a set $\{y \mid \text{there exists } x \text{ such that } (x, y) \text{ has property } P\}$.

As with Axiom **??**, mathematicians regard Axiom **??** as being an infinity of axioms, one for each replacement property.

2.9 The Existence of a Set

So far we have defined sets with postulates, derived some of their properties, and stated five axioms which guarantee that if a set exists then so do all of its relatives. We have not yet declared the existence of a set in our logical system. There is a risk inherent in making such a declaration. It is possible that the set postulates and the axioms of the basic set operations conflict in some way. It may be that it is impossible for anything to satisfy all the postulates and axioms of sets. Just because we write down some postulates and axioms does not mean it is possible for something to satisfy them. If we define a 'svevnerblin' to be a thing which is unequal to itself, then, because of Axiom **??**, declaring the existence of a svevnerblin in our logical system would make our basis set logically contradictory. There is a possiblility that the same will happen if we declare the existence of a set in our logical system.

When we declare the existence of a set, it is not possible for us to prove that this declaration has not made our basis set logically contradictory. To do so we would have to go through every one of the set's relatives and show that each obeyed the set postulates. This would take an infinite amount of time. However, a large number of mathematicians have worked with the assumption that a set exists, and none of them have found fault with it, so we shall declare the existence of a set in our own logical system and trust that our basis set will not be logically contradictory as a result of our assumption.

Axiom 2.25 (Weak Existence) There exists a set.

We call this axiom the Weak Axiom of Existence because it does not specify the members of the set — it merely says there exists *a* set. We do not know what the set's members are. This may seem useless, but it is not, for it allows us to prove the existence of the empty set.

Theorem 2.26 (Existence of the Empty Set) There exists a set with no members.

Proof of Theorem 2.26 Axiom **??** and Axiom **??** together imply there exists a property, Q, such that x has property Q if and only if $x \neq x$. Axiom **??** states that there exists a set. Call it A. A is a set, so Axiom **??** states that $B = \{x \mid x \in A \text{ and } x \text{ has property } Q\} = \{x \mid x \in A \text{ and } x \neq x\}$ exists. By Axiom **??** there is no x such that $x \neq x$, so B is a set with no members. □

The empty set exists in our logical system, so all its relatives exist as well. Many mathematical tools can be constructed out of the empty set, so many of the tools of pure mathematics already exist in our logical system. Because the empty set is so important in mathematics, we shall take time to prove a couple of theorems about it.

Theorem 2.27 (Uniqueness of the Empty Set) Let A and B be sets with no members. Then $A = B$.

Proof of Theorem 2.27 Neither A nor B have any members, so there is no member of A which is not a member of B and there is no member of B which is not a member of A, so Postulate **??** states that $A = B$. □

There are several symbols for the empty set. I have already used $\{\}$. Here is another one:

Definition 2.28 (A Symbol for the Empty Set) The set with no members is denoted $\emptyset$.

Here is the second theorem about the empty set.

Theorem 2.29 If C is any set then $\emptyset \subset C$.

Proof of Theorem 2.29 Suppose C is a set. $\emptyset$ has no members, so there is no member of $\emptyset$ which is not a member of C, so by Definition **??** we have $\emptyset \subset C$. □

Here is an important theorem about subsets. It makes use of Theorem **??**.

Theorem 2.30 (Transitive Inclusion) If A, B, and C are sets such that $A \subset B$ and $B \subset C$, then $A \subset C$.

Proof of Theorem 2.30 Suppose A, B, and C are sets such that $A \subset B$ and $B \subset C$. If $A = \emptyset$ then Theorem **??** states that $A \subset C$. If $A \neq \emptyset$ then let x be any member of A. $A \subset B$, so $x \in B$. $B \subset C$, so $x \in C$. We chose x arbitrarily in A, so all $x \in A$ are members of C, so there is no member of A which is not a member of C, so $A \subset C$. □

2.10 The Natural Numbers

Numbers are constructed out of the empty set in the following way:

Definition 2.31 (Numerals) We define $0 = \emptyset$, $1 = \{\emptyset\} = \{0\} = 0 \cup \{0\}$, $2 = \{\emptyset,\ \{\emptyset\}\} = \{0, 1\} = 1 \cup \{1\}$, $3 = \{\emptyset,\ \{\emptyset\},\ \{\emptyset,\ \{\emptyset\}\}\ \} = \{0, 1, 2\} = 2 \cup \{2\}$, ... If n is any numeral, then $n + 1$ is the numeral which follows n in the sequence of numerals. For every numeral, n, we have $n + 1 = n \cup \{n\}$.

If we equate the meaning of the phrase 'less than' to 'is a member of', then we see 0 is less than 1 because $0 \in 1$, and 1 is less than 3 because $1 \in 3$.

The 'natural numbers' are the numbers 1, 2, 3, 4,... The set of natural numbers is therefore $\{1, 2, 3, \ldots\}$. It is represented by the symbol '$\mathbb{N}$'. Thus $\mathbb{N}$ is an infinite set. It has infinitely many members. We cannot prove $\mathbb{N}$ exists even with the help of Axiom **??**, because it takes *infinitely*[5] many basic set operations to make $\mathbb{N}$ out of $\emptyset$, so any demonstration of the existence of $\mathbb{N}$ in our logical system would be infinitely long. However, mathematicians have never run into any logical contradiction by declaring the existence of $\mathbb{N}$ in their logical systems, so we shall feel free to follow their example as we did with the Weak Axiom of Existence.

Axiom 2.32 (Infinity) There exists a set, $\mathbb{N}$, such that $\mathbb{N} = \{1, 2, 3, \ldots\}$.

If we can replace every member of a set, A, with a member of $\mathbb{N}$, without ever using the same natural number twice, then we say A has 'countably' many members. If this cannot be done, then we say A has 'uncountably' many members. Clearly the set $\mathbb{N}$ has countably many members because we can replace every member of $\mathbb{N}$ with itself; but the power set of $\mathbb{N}$, denoted $\wp(\mathbb{N})$, whose existence is implied by Axiom **??**, does not have countably many members. It has uncountably many members.

Using the natural numbers, we can construct the rational numbers and the real numbers (π, $\sqrt{2}$, as well as real number equivalents to 1, 2, and 3). A real number is a complicated set, but Axiom **??** can be used to prove its existence in our logical system.

The Weak Axiom of Existence is a logical concequence of the Axiom of Infinity, so the Axiom of Infinity makes the Weak Axiom of Existence redundant.

[5] 'Infinitely many' means 'never endingly many'.

2.11 Mappings

Mappings are what mathematicians use to represent relationships between sets. If $S = \{s \mid s \text{ is a ship on the sea}\}$, and $P = \{p \mid p \text{ is a position on the surface of the Earth}\}$, then each member of S is paired with exactly one member of P. The whereabouts of all ships may be represented by a set of ordered pairs with the following properties:
(1) There is one and only one ordered pair for each ship.
(2) The first member of that ordered pair is the ship itself.
(3) The second member of that ordered pair is the ship's position.
This set of ordered pairs is an example of a 'mapping'. We shall call it W, the 'whereabouts mapping' for ships. The wherabouts mapping is a mapping 'from' the set of ships 'to' the set of positions. It represents the relationship between ships and positions.

Definition 2.33 (Mappings) If A and B are sets, then a set M is a 'mapping from A to B' if and only if it has the following properties:
(1) Every member of M is an ordered pair (x, y) where $x \in A$ and $y \in B$.
(2) For every $x \in A$ there is one and only one $y \in B$ for which $(x, y) \in M$.

We can use Axiom **??** to prove the existence of mappings between existing sets.

Theorem 2.34 (Existence of Mappings) If A and B are sets, and M is a mapping from A to B, then M exists.

Proof of Theorem 2.34 Suppose A and B are sets, and M is a mapping from A to B. Let P be the property of being some $(x,\ (x, y))$ for which $x \in A$ and $(x, y) \in M$. Definition **??** implies that for every $x \in A$ there is a unique (x, y) such that $(x,\ (x, y))$ has property P. Thus Axiom **??** implies P exists. A is a set, so Axiom **??** implies $\{(x, y) \mid (x,\ (x, y)) \text{ has property } P \text{ for some } x \in A\} = M$ exists. □

In the example of ships and positions, every ship has a position, but not every position has a ship. The set of positions which have ships is called the 'range' of W, while S is called the 'domain' of W. These and a few more terms are defined formally in the next paragraph:

Definition 2.35 (Features of Mappings) Let A and B be sets, and let M be a mapping from A to B. We say A is the *domain of* M. We say B is the *target of* M. If $C \subset A$, then the *image of* C *in* M, denoted $M(C)$, is $\{y \mid \text{there exists } x \in C \text{ such that } (x, y) \in M\}$. $M(A)$ is called the *range of* M. If $x \in A$, then the *value of* M *at* x, denoted $M(x)$, is the unique $y \in B$ such that $(x, y) \in M$.

If s is a ship on the sea, then $W(s)$ is the position of s. If the set of all ships exists, and the set of all positions exists, does this mean the set of all ship positions, $W(S)$, exists? We shall use Axiom **??** to prove this is so.

Theorem 2.36 (Existence of Images) If A and B are sets, and M is a mapping from A to B, and C is a set such that $C \subset A$, then $M(C)$ exists.

Proof of Theorem 2.36 Suppose A and B are sets, M is a mapping from A to B, and C is a set such that $C \subset A$. M is a mapping, so Definition **??** implies that for every $x \in C$ there is a unique y such that $(x, y) \in M$. C exists, so Axiom **??** implies $\{y \mid \text{there exists } x \in C \text{ such that } (x, y) \in M\} = M(C)$ exists. □

Many ships on the sea are moving, so W is accurate only for a single instant in time. Let us define a more sophisticated mapping to represent the movements of all ships at all times. We shall call this mapping X. We would like X to tell us the position of any ship at any point in time. Let $T = \{t \mid t \text{ is a point in time }\}$. Remembering that S is the set of all ships at sea, we see that a ship together with a point in time may be represented by and ordered pair (s, t), where $s \in S$ and $t \in T$. We define $H = \{(s, t) \mid s \in S \text{ and } t \in T\}$. Each member of H specifies a ship and a point in time.

Before we define X, there is a complication we must sort out. The Thermopilae was a tea clipper destroyed at the beginning of the twentieth century, so there is no position on the globe corresponding to the member of H defined by: (Thermopilae, 18:09:20 5 June 1991). To get around this problem, we extend P to include a member which corresponds to ships no longer on the seas or not yet on the seas. Let $P' = P \cup \{ \text{'nowhere'} \}$. The word 'nowhere' is a member of P'. We pair the word 'nowhere' with any member of H which represents a ship at a time before it was made or after it was destroyed. We pick 'nowhere' to represent the position of (Thermopilae, 18:09:20 5$^{\text{th}}$ June 1991).

Now we can define X. Let X be a mapping from H to P' such that the value of X at $(s, t) \in H$ is the position of ship s at time t. If the ship is sailing on the sea at time t, then the position is (x, y), where x is its latitude and y is its longitude. Otherwise the postition is 'nowhere'. The members of X are ordered pairs $((s, t),\ p)$ were $p \in P'$.

2.12 Final Remark

The set P defined above represents all points on the surface of the globe. The members of P are ordered pairs, (x, y), of latitude and longitude. The ordered pair (x, y) is the set $\{\{x\}, \{x, y\}\}$. The latitude, x, and the longitude, y, are represented by real numbers. Real numbers are constructed out of the members of $\mathbb{N}$ using the basic set operations. The natural numbers are in turn constructed out of the empty set using the basic set operations. This means that, although the set P is a working representation of the surface of the globe, we find when we examine its contents, and dive deep within the curly brackets which define it, we come ultimately to: nothing! This is one of the beauties of mathematics: we use nothing to represent something, and so offend no one.

Chapter 3

Frames

A frame is a philosophical tool just as a mapping is a mathematical tool. After an introduction to set theory, we know a mapping is a particular type of set. The same is true for frames. A frame is a particular type of set. It is wholly abstract. It is a tool designed to describe systems of thought. Logic is an example of a system of thought. There are an infinity of others. Each can be described by a frame. The words 'contradiction', 'assertion', 'equivalency', 'truth' and 'falsehood' are all clearly defined within each frame. We shall not define these terms in any absolute sense. Their meaning is tied to the nature of the frame in which they are used.

3.1 Introducing Frames

In the previous chapter, we began the construction of a logical system. The language we use in the system is Restricted Mathematical English. Let $\mathcal{E}_L$ be the set of all statements in this language. By 'statement' we mean any sequence of syntactically and grammatically correct sentences. The subscript 'L' is used to mark out $\mathcal{E}_L$ as the language of our Logical system of reasoning. The statement, *'for every x we have $x = x$'* is a member of $\mathcal{E}_L$; which is to say, Axiom **??** $\in \mathcal{E}_L$. Likewise, every other formal definition, axiom or postulate we have declared so far is an element of $\mathcal{E}_L$. Thus $\Delta \subset \mathcal{E}_L$.

The logical consequences of Δ are also a subset of $\mathcal{E}_L$. Given any $A \subset \mathcal{E}_L$, there is a set $B \subset \mathcal{E}_L$ which is the set of all logical consequences of A.[1] Every $A \subset \mathcal{E}_L$ is paired by the rules of logic to a unique $B \subset \mathcal{E}_L$. We recall that the power set of $\mathcal{E}_L$ (denoted $\wp(\mathcal{E}_L)$) is defined as the set of all subsets of $\mathcal{E}_L$. Thus we see that every member of $\wp(\mathcal{E}_L)$ is paired with another member of $\wp(\mathcal{E}_L)$. This relationship between members of $\wp(\mathcal{E}_L)$ may be represented by a mapping from $\wp(\mathcal{E}_L)$ to $\wp(\mathcal{E}_L)$. The target of this mapping is the same as its domain. Let us denote this mapping by the symbol $\mathcal{T}_L$. If $A \in \wp(\mathcal{E}_L)$ (that is, $A \subset \mathcal{E}_L$), then $\mathcal{T}_L(A)$ is the set of all logical consequences of A. The set $\mathcal{T}_L(A)$ is a subset of $\mathcal{E}_L$, which is to say it is a member of $\wp(\mathcal{E}_L)$.

Let $\mathcal{L}_L$ be the set of all statements in $\mathcal{E}_L$ which are *not* of the form '$A\overline{A}$'. From Chapter 2 we recall that a set of statements is said to be logically contradictory if and only if it has a logical consequence of the form '$A\overline{A}$'. A set of statements is logically contradictory if and only if it has a logical consequence which is *not* a member of $\mathcal{L}_L$. In terms of $\mathcal{T}_L$, this means $A \subset \mathcal{E}_L$ is logically contradictory if and only if there is a member of $\mathcal{T}_L(A)$ which is not a member of $\mathcal{L}_L$. By Definition **??**, this is the same as saying A is logically contradictory if and only if $\mathcal{T}_L(A) \not\subset \mathcal{L}_L$ (the symbol '$\not\subset$' means 'is not a subset of').

The sets $\mathcal{E}_L$, $\mathcal{L}_L$, and $\mathcal{T}_L$ provide us with a complete description of the behaviour of our logical system of reasoning. They make no mention of the logical steps, but the effects of these steps are represented by

[1]This statement reads 'Given any A, a subset of $\mathcal{E}_L$, there is a set B, a subset of $\mathcal{E}_L$, which is the set of all logical consequences of A.'

$\mathcal{T}_L$. We shall find that any system of reasoning may be represented by a 'domain' ($\mathcal{E}$), a 'legal subset' of the domain ($\mathcal{L}$), and a 'consequence mapping' ($\mathcal{T}$). A frame is the combination of these three things into an ordered triple:

Definition 3.1 (Frames) A *frame* is any ordered triple $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ where $\mathcal{E}$, $\mathcal{L}$, and $\mathcal{T}$ are sets such that $\mathcal{L} \subset \mathcal{E}$ and $\mathcal{T}$ is a mapping from $\wp(\mathcal{E})$ to $\wp(\mathcal{E})$.

A frame is a system in which sets are paired with other sets according to a pattern defined by the consequence mapping. We can visualize the domain of a frame as a flat rectangular shape. Points inside the borders of the rectangle are members of the domain. We think of subsets of the domain as shapes drawn on the surface. Such shapes enclose some points and exclude others. The points inside a shape are the members of the subset. The legal subset of the frame can be seen as a shape (perhaps an oval) drawn in the rectangle representing the domain. The consequence mapping pairs shapes with other shapes, or collections of points with other collections of points. This visualization allows us to draw sketches representing frames upon pieces of paper.

Let $\mathcal{F}_L = (\mathcal{E}_L, \mathcal{L}_L, \mathcal{T}_L)$ be the frame which represents our logical system of reasoning. We call it the 'Frame of Restricted Mathematical English and Logic'. If $A \subset \mathcal{E}_L$ then $\mathcal{T}_L(A)$ is the set of all logical consequences of A. By adding an axiom to Δ, we could declare the existence of $\mathcal{F}_L$ in our logical system. Because there are no statements akin to, *'This statement is not a theorem,'* in $\mathcal{E}_L$, and because we do not allow ourselves to be mislead by improper use of symbols (see p **??**), we shall not arrive at any contradiction by declaring the existence of $\mathcal{F}_L$ within our logical system. We could also declare the existence of all the other frames I shall define in this book. Even without declaring the existence of $\mathcal{F}_L$ in our logical system, there already exist some frames. For example, from Theorem **??** (p **??**), we can prove that the frame $\mathcal{F} = (\emptyset, \emptyset, \{(\emptyset, \emptyset)\})$ exists in our logical system.

3.2 Contradiction in a Frame

The next formal definition defines the phrase 'contradictory in a frame'. In the last chapter, I was careful to say 'logically contradictory' rather than just 'contradictory', because contradiction is subjective. Its meaning depends upon the system of thought one uses. If we were to change the rules of logic, then it is easy to imagine that some sets of statements which were not contradictory before the rule changes would be contradictory after the rule changes. We could add the rule of proof by analogy to the logical steps. Or we could disallow the use of the rule of extraction.

Let us create a new frame out of $\mathcal{F}_L$ by changing its consequence mapping. The new consequence mapping shall be called $\mathcal{T}_{\text{dragon}}$. It is such that for every $A \subset \mathcal{E}_L$, the only member of $\mathcal{T}_{\text{dragon}}(A)$ is the statement, *'Dragons once roamed the world.'* We shall call this frame the 'dragon frame', and denote it $\mathcal{F}_{\text{dragon}}$, where $\mathcal{F}_{\text{dragon}} = (\mathcal{E}_L, \mathcal{L}_L, \mathcal{T}_{\text{dragon}})$. Thus for every $A \subset \mathcal{E}_L$ we have $\mathcal{T}_{\text{dragon}}(A) = \{$*'Dragons once roamed the world'*$\}$. If we use $\mathcal{F}_{\text{dragon}}$ to guide our reasoning, then regardless of what axioms we start off with, we shall always conclude that, *'Dragons once roamed the world.'* In $\mathcal{F}_{\text{dragon}}$ it is impossible to become entangled with any form of paradox or contradiction. Our axioms make no difference to our conclusion. We can never come up with conflicting conclusions in $\mathcal{F}_{\text{dragon}}$. Not even a set of axioms which is *logically* contradictory will give us any problems in $\mathcal{F}_{\text{dragon}}$.

'Contradiction' has different meanings in different frames. If we were to pick one frame above all others as the 'best' frame, and define contradiction in some absolute sense with respect to this frame, then we would be guilty of favouritism. Favouritism is what I have tried to steer clear of in the design of my philosophical tools. I do not want the tools to have any inherent prejudice against even systems of thought as ridiculous as the one about dragons. After all, it is entirely possible that someone will go mad and use the dragon reasoning as their guide in life. Then we could describe the symptoms of their insanity with a frame. Or perhaps believing in dragons is the most important thing in life; so it doesn't matter why we believe, but only that we do believe.

For this reason, we define contradiction in the following way:

Definition 3.2 (Contradiction in a Frame) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame and $A \subset \mathcal{E}$, then we say A is *contradictory in* $\mathcal{F}$ if and only if $\mathcal{T}(A) \not\subset \mathcal{L}$.

Now, instead of saying 'logically contradictory', we can say 'contradictory in $\mathcal{F}_L$'. No set of statements is contradictory in the dragon frame, since the only statement we ever come up with in the dragon frame is, *'Dragons once roamed the world.'* This does not have the form '$A\overline{A}$', so it is a member of of $\mathcal{L}_L$, which is the legal subset of $\mathcal{F}_{\text{dragon}}$ — just as it is the legal subset of $\mathcal{F}_L$.

The meaning of the word 'contradiction', as we use it in the above definition, is in keeping with the way the word is used in the phrase 'contradictory evidence'. The word is used in a different sense when we say 'these two facts are contradictory to one another'. We shall support this second use of the word with the following definition:

Definition 3.3 (Special Case of Contradiction) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, and x and y are members of $\mathcal{E}$, then we say x is *contradictory in* $\mathcal{F}$ *to* y if and only if $\{x, y\}$ is contradictory in $\mathcal{F}$.

Thus, *'There exists a brontasaurus,'* is contradictory in $\mathcal{F}_L$ to, *'There does not exist a brontasaurus.'*

3.3 The Mind as a Frame

The Frame of Restricted Mathematical English and Logic is an example of a frame used for communication. It is not necessarily the same as or even similar to the workings of our minds. But our minds may also be represented by frames. In general, each mind will be represented by a different frame. Given a mind, we can choose a domain, a legal subset, and a consequence mapping which together make up a frame representing the mind.

Let us first define the domain of the frame representing the mind. We shall call it $\mathcal{E}_M$. The subscript M indicates the mind we are trying to represent. The members of $\mathcal{E}_M$ are thoughts in the mind M. This mind need not be human. It could be the mind of a god or a computer. If M were a human mind, then every $x \in \mathcal{E}_M$ would be a sequence of electrical signals passing between the neurons of the brain. If M were a computer, then every $x \in \mathcal{E}_M$ would be a sequence of electrical signals passing through the registers of its central processor. The only thing we require of $\mathcal{E}_M$ is that it include everything the mind might ever think. Thus we define $\mathcal{E}_M$ to be the set of all thoughts which it is possible for a mind such as M to think.

For the legal subset, $\mathcal{L}_M$, we choose the set of all concepts which the mind does not regard as impossible. While most people are convinced that, for example, human beings can only be in one place at a time, there are still many matters of impossibility upon which people disagree. One person might find the concept of a saintly mass murderer impossible, while another might find it entirely plausable. If two such people were to argue about the saintliness of particular mass murderer, then they would do best if they were aware of this difference between their minds — else they might not get the most out of their discussion.

As time goes by, the set of things which each of us regard as impossible is likely to change. Such changes would be part of the process we call 'learning'. In the light of our experiences, we might find we change our minds about the plausibility of many things. The frame used to represent somebody's mind on Tuesday may not be an accurate representation of that person's mind on Thursday.

The next thing we do is define a consequence mapping for the frame. We call it $\mathcal{T}_M$. It is a mapping from $\wp(\mathcal{E}_M)$ to $\wp(\mathcal{E}_M)$. The mapping $\mathcal{T}_M$ represents a personal system of reasoning in the same way $\mathcal{T}_L$

represents a logical system of reasoning. There are many ways we could define $\mathcal{T}_M$. The one I present in the following paragraph is just one example.

We start by letting A represent any set of concepts. We define $\mathcal{T}_M(A)$ to be the set of all concepts which this mind decides are consequences of A after it has been allowed exactly one minute of uninterrupted contemplation. The set A might represent the mind's knowledge of the evidence in a legal case. Then $\mathcal{T}_M(A)$ would represent the mind's conclusions on the case after it has thought about the evidence for one minute.

Now can we represent my mind with a frame $\mathcal{F}_A$, where 'A' is for Author, and your mind with a frame $\mathcal{F}_R$, where 'R' is for Reader. We shall refer to these two frames often.

3.4 Moving Between Frames

I am using the Frame of Restricted Mathematical English and Logic to communicate the properties of my philosophical tools. I use statements in $\mathcal{L}_L$ to represent my axioms. I use $\mathcal{F}_L$ to represent the properties of sets: the logical consequences of the set postulates represent the properties. As you read the statements in this book, they bring concepts to your mind. The only test I can make of the accuracy of this communication (and it is by no means a conclusive test) is to ask you to answer questions and see how you respond. If your responses are not in keeping with the way I would to answer the same questions, then I can assume there is a misunderstanding.

Therefore, in this book, we rely upon the accuracy of two moves between frames. The first is from $\mathcal{F}_A$ to $\mathcal{F}_L$, and the second is from $\mathcal{F}_L$ to $\mathcal{F}_R$. In the next chapter, we shall represent the properties of such moves with another philosophical tool called the 'symbolism'. Until then, we shall keep these moves in mind, but we shall not delve any deeper into the problem of describing them.

3.5 Proper Frames

By examining the properties of these three elements in a frame, it is possible to define classes of frames, and we shall find it is useful to do so. The first class of frames we shall define is that of 'proper' frames:

Definition 3.4 (Proper Frames) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, then we say $\mathcal{F}$ is *proper* if and only if it has the following properties:
(1) For every $A \subset \mathcal{E}$ we have $A \subset \mathcal{T}(A)$.
(2) For every A and B such that $A \subset B \subset \mathcal{E}$ we have $\mathcal{T}(A) \subset \mathcal{T}(B)$.
(3) $\emptyset$ is not contradictory in $\mathcal{F}$.

We recall that in logical reasoning, the first things we label as logical consequences of a basis set are the members of the basis set itself. That is to say, if A is our basis set, then every member of A is a member of $\mathcal{T}_L(A)$. Thus $A \subset \mathcal{T}_L(A)$. In the dragon frame, it is not true to say that for all $A \subset \mathcal{E}_L$ we have $A \subset \mathcal{T}_{\text{dragon}}(A)$, since for any $A \subset \mathcal{E}_L$, $\mathcal{T}_{\text{dragon}}(A)$ contains only the statement, *'Dragons once roamed the world.'* Thus, because of property (1) of Definition **??**, the dragon frame is not proper.

If we add statements to the basis set in a logical system, we are sure that every logical consequence of the previous basis set will still be a logical consequence of the expanded basis set. This need not be true for all frames. For example, let us define a frame $\mathcal{F}_W = (\mathcal{E}_L, \mathcal{L}_L, \mathcal{T}_W)$ such that for every $A \subset \mathcal{E}_L$ we have $\mathcal{T}_W(A)$ is the set of all members of $\mathcal{E}_L$ which are *not* members of A. Thus $\mathcal{T}_W(\emptyset) = \mathcal{E}_L$, and $\mathcal{T}_W(\mathcal{E}_L) = \emptyset$. To add to A is to subtract from $\mathcal{T}_W(A)$, so, because of property (2) of Definition **??**, $\mathcal{F}_W$ is not proper.

The last property of a proper frame is that the empty set is not contradictory in the frame. If the empty set were contradictory in the frame, and the frame had property (2), then every subset of the domain would be contradictory in the frame.[2]

The Frame of Restricted Mathematical English and Logic is proper. Perhaps, upon reflection, you might decide that your own mind is proper too. It is no shame if it is not. To be ashamed that one's mind is not 'proper' would be to take my choice of words too seriously. I could equally well have used the word 'wibble' to mean 'proper', and no meaning would have been lost.

Here is a theorem stating an important property of proper frames.

Theorem 3.5 If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a proper frame, and A and B are sets such that $A \subset B \subset \mathcal{E}$, then B is contradictory in $\mathcal{F}$ if A is contradictory in $\mathcal{F}$.

This theorem means that in a proper frame it is impossible to add a statement to a contradictory set and get a set which is not contradictory in the frame. The only way of resolving a contradiction is by removing statements from the set. Suppose we agree to use $\mathcal{F}_L$ as a frame in which to conduct a discussion about the validity of arguments presented in a book. We know $\mathcal{F}_L$ is proper, so if we find after the first chapter that the statements made by the author are contradictory in $\mathcal{F}_L$, then we know that nothing he says thereafter in the book can resolve the contradiction. His basis set will always be contradictory in $\mathcal{F}_L$.

Here is a proof of Theorem **??**:

Proof of Theorem 3.5 Using the symbols defined in the thoerem, suppose A is contradictory in $\mathcal{F}$. Definition **??** states that $\mathcal{T}(A) \not\subset \mathcal{L}$, so let us pick some $x \in \mathcal{T}(A)$ such that $x \notin \mathcal{L}$. $A \subset B$ and $\mathcal{F}$ is proper, so Definition **??** implies $\mathcal{T}(A) \subset \mathcal{T}(B)$, so $x \in \mathcal{T}(B)$, so $\mathcal{T}(B) \not\subset \mathcal{L}$, so Definition **??** states that B is contradictory in $\mathcal{F}$. □

3.6 Consequences in a Frame

We have defined the phrase 'contradictory in a frame'. We see that what is contradictory in your mind (described by the frame $\mathcal{F}_R$) need not be contradictory in my mind (described by the frame $\mathcal{F}_A$) or in the Frame of Restricted Mathematical English and Logic (denoted $\mathcal{F}_L$). Contradiction is defined with respect to a frame.

For the sake of argument, suppose we had defined the word 'contradictory' with a statement, 'A contradictory basis set is one which both implies and denies the truth of the same fact'. This definition relies upon the meanings of the words 'truth', 'implies', and 'denies'. It would be interpreted by each of us to mean that a contradictory set of statements is one for which the set of concepts it represents is contradictory in our minds. But everyone's mind is different in this respect, so the definition is subjective. If we were unaware of its inherent subjectivity, then we would not be conscious of that subjectivity when we spoke to other people. Our discussions with them about sensitive ethical issues would tend to be futile. We would overloook the fact that others need not think as we do. What is to us irrefutable evidence in support of our opinion may to them be not at all convincing.

I have qualified my definition of the word 'contradiction' with the phrase 'in a frame'. I shall do the same for the words 'truth', 'falsehood', 'equivalent', 'assertion', and 'consequence'. The first of these to be defined is 'consequence'.

Definition 3.6 (Consequence in a Frame) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, and $A \subset \mathcal{E}$, then any $x \in \mathcal{T}(A)$ is said to be a *consequence in $\mathcal{F}$ of A*.

Now we can replace the phrase 'logical consequence' with 'consequence in $\mathcal{F}_L$', just as we can replace 'logically contradictory' with 'contradictory in $\mathcal{F}_L$'. As with contradiction, we shall define a special case of consequence in a frame:

[2]For proof of this statement, see Theorem **??**

Definition 3.7 (Special Case of Consequence) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, and x and y are elememts of $\mathcal{E}$, then we say x is a *consequence in* $\mathcal{F}$ *of* y if and only if x is a consequence in $\mathcal{F}$ of $\{y\}$.

Thus, *'There exists a brontasaurus,'* is a consequence in $\mathcal{F}_L$ of, *'The largest brontasaurus lives in Montana.'* In the dragon frame, we have, *'Dragons once roamed the world,'* as the only consequence in $\mathcal{F}_{\text{dragon}}$ of, *'The largest brontasaurus lives in Montana.'*

3.7 Paradoxes in Frames

I shall define a 'paradox in a frame' to be a member of the domain which is contradictory to itself.

Definition 3.8 (Paradox in a Frame) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, then we say $x \in E$ is a *paradox in* $\mathcal{F}$ if and only if x is contradictory in $\mathcal{F}$ to x.

We can now say 'paradox in $\mathcal{F}_L$' instead of 'logical paradox'. The statement, *'It is raining and it is not raining,'* is a paradox in $\mathcal{F}_L$. However, it is not a paradox in $\mathcal{F}_{\text{dragon}}$, since nothing is contradictory in $\mathcal{F}_{\text{dragon}}$.

3.8 Assertion in a Frame

Here we define the meaning of 'assertion' within a frame:

Definition 3.9 (Assertion in a Frame) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, and $A \subset \mathcal{E}$, and $x \in \mathcal{E}$, then we say x is an *assertion in* $\mathcal{F}$ *of* A if and only if x and A have the following properties:
(1) x is not a paradox in $\mathcal{F}$.
(2) A is not contradictory in $\mathcal{F}$.
(3) $\{x\} \cup A$ is not contradictory in $\mathcal{F}$.
(4) For every $y \in \mathcal{E}$ which is contradictory in $\mathcal{F}$ to x we have $\{y\} \cup A$ is contradictory in $\mathcal{F}$.

Every statement in $\mathcal{E}_L$ is an assertion in $\mathcal{F}_{\text{dragon}}$ of every $A \subset \mathcal{E}_L$. Let us consider whether the same may be said for $\mathcal{F}_L$. Suppose $x \in \mathcal{E}_L$ is an assertion in $\mathcal{F}_L$ of some $A \subset \mathcal{E}_L$. If there is a logical consequence of A which means the same thing as x, then the rule of logical equivalency states that x is also a logical consequence of A. If there is no consequence in $\mathcal{F}_L$ of A which means the same thing as x, then there is some aspect of the meaning of x which is not expressed by the logical consequences of A. We assume that Restricted Mathematical English contains a statement, y, expressing only that part of the meaning of x which is not implied by the logical consequences of A. Then we can always find a negation of y, denoted $\overline{y}$, which is contradictory in $\mathcal{F}_L$ to x, but is such that $\{\overline{y}\} \cup A$ is not contradictory in $\mathcal{F}_L$ — in which case, x is not an assertion in $\mathcal{F}_L$ of A. On the strength of our assumption about the statements available in Restricted Mathematical English, x must be a consequence in $\mathcal{F}_L$ of A if it is to be an assertion in $\mathcal{F}_L$ of A; so every assertion in $\mathcal{F}_L$ of A is also a consequence in $\mathcal{F}_L$ of A.

We have a special case of assertion between two members of $\mathcal{E}$, just as we had a special case of contradiction between two members of $\mathcal{E}$.

Definition 3.10 (Special Case of Assertion) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, and x and y are members of $\mathcal{E}$, then we say x is an *assertion in* $\mathcal{F}$ *of* y if and only if x is an assertion in $\mathcal{F}$ of $\{y\}$.

The statement, *'There exists a living brontasaurus,'* is an assertion in $\mathcal{F}_L$ of, *'The weight of the largest living brontasaurus is 500 tons,'* because any statement contradictory in $\mathcal{F}_L$ to the first statement is also contradictory in $\mathcal{F}_L$ to the second statement. However, note that the second statement is not an assertion in $\mathcal{F}_L$ of the first statement, since not every statement contradictory in $\mathcal{F}_L$ to, *'The largest living brontasaurus weighs 500 tons,'* is contradictory in $\mathcal{F}_L$ to, *'There exists a living brontasaurus,'* (consider, for example, the statement, *'The largest living brontasaurus weighs 501 tons.'*)

3.9 Thorough Frames

To be considered 'thorough', a frame must have two properties. The first property of a thorough frame is that every assertion in the frame of a basis set is also a consequence in the frame of the basis set. It is easy enough to think up frames in which this is not the case. I do not think it is the case in $\mathcal{F}_A$, but I leave it to you to decide whether it is the case in $\mathcal{F}_R$.

Let us use $\mathcal{F}_L$ to illustrate the second property of thorough frames. Let $A \subset \mathcal{E}_L$. Then $\mathcal{T}_L(A)$ is the set of consequences in $\mathcal{F}_L$ of A. Let $C = \mathcal{T}_L(A)$. What is the set of consequences in $\mathcal{F}_L$ of C? That is, what is $\mathcal{T}_L(C) = \mathcal{T}_L(\mathcal{T}_L(A))$? When we look at the way the logical steps work, we see that $\mathcal{T}_L(C) = C$. Thus we have $\mathcal{T}_L(\mathcal{T}_L(A)) = \mathcal{T}_L(A)$. This is the second property of a thorough frame: that for all $A \subset E$ we have $\mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A)$.

The more time we allow ourselves to think about something, the more consequences we are likely to come up with. Thus frames representing our minds are unlikely to be thorough. Certainly the one-minute representations defined above ($\mathcal{F}_A$ and $\mathcal{F}_R$) are not thorough.

Definition 3.11 (Thorough Frames) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, then $\mathcal{F}$ is *thorough* if and only if for every $A \subset \mathcal{E}$ we have:
(1) Every assertion in $\mathcal{F}$ of A is a consequence in $\mathcal{F}$ of A.
(2) $\mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A)$.

3.10 Equivalence in a Frame

We shall now define what it is for two sets to be equivalent in a frame.

Definition 3.12 (Equivalence in a Frame) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, and A and B are subsets of $\mathcal{E}$, then we say A is *equivalent in* $\mathcal{F}$ to B if and only if $\mathcal{T}(A) = \mathcal{T}(B)$.

Every set of statements is equivalent in $\mathcal{F}_{\text{dragon}}$ to every other set of statement, since every set of statements has the same consequences in $\mathcal{F}_{\text{dragon}}$.

Now for the inevitable 'special case of equivalency'.

Definition 3.13 (Special Case of Equivalency) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, and x and y are members of $\mathcal{E}$, then we say x is *equivalent in* $\mathcal{F}$ to y if and only if $\{x\}$ is equivalent in $\mathcal{F}$ to $\{y\}$.

The statement *'For every x we have $x = x$'* is equivalent in $\mathcal{F}_L$ to the statement *'There is no x such that $x \neq x$.'* Here is a theorem which applies to all frames.

Theorem 3.14 (Self-Equivalency of Members of $\mathcal{E}$) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, and $x \in \mathcal{E}$, then x is equivalent in $\mathcal{F}$ to itself.

Proof of Theorem 3.14 Using the symbols defined in the theorem, we see that, by Postulate **??** we have $\mathcal{T}(\{x\}) = \mathcal{T}(\{x\})$, so by Definition **??** we have x equivalent in $\mathcal{F}$ to itself.

3.11 Truth and Falsehood in a Frame

We have definitions for assertion, equivalence, consequence, paradox, and contradiction in a frame. We have proved a few theorems. We have become more familiar with the way frames may be used to describe systems of thought. I shall now offer definitions for the phrases 'truth in a frame' and 'falsehood in a frame'. The words 'truth' and 'falsehood' are laden with portent. Many books have been written about the pursuit of truth. In these books we find passages akin to the following:

'Truth: what is it? If we were to find it, how would we know we had found it? If we know for certain what it is like, then surely this knowledge itself is truth? So, we must already have some of it if we are to ever find more of it... '

Sounds clever, but it is merely a bewitchment of our intelligence by language.[3] Authors of such books usually insist that the word 'truth' cannot be defined, or they spend a long time building up to their own personal definition of the word — which is what I am doing. However, I shall not define the word 'truth' in any absolute sense. I shall define the phrase 'true in $\mathcal{F}$ with respect to A'. My definition tells us nothing about the universe. It only attaches a word to a particular relationship between things in a frame, and a frame is completely abstract.

Definition 3.15 (Truth in a Frame) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, and $A \subset \mathcal{E}$, and $x \in \mathcal{E}$, then we say x is *true in* $\mathcal{F}$ *with respect to* A if and only if A and x have the following properties:
(1) x is not a paradox in $\mathcal{F}$.
(2) A is not contradictory in $\mathcal{F}$.
(3) x is a consequence in $\mathcal{F}$ of A.

Theorem **??** is true in $\mathcal{F}_L$ with respect to Δ — so long as we are correct in our assumption that Δ is not contradictory in $\mathcal{F}_L$. In the dragon frame, the statement, *'Dragons once roamed the world,'* is true with respect to all $A \subset \mathcal{E}_L$. The statement, *'Some apples are red,'* is *not* true in $\mathcal{F}_L$ with respect to Δ — although it may be true in $\mathcal{F}_R$ with respect to your assumptions about apples. Nothing in Δ leads us to any logical conclusions about apples.

If a statement is not true, does that mean it is false? If we were to add the statement, *'Some apples are red,'* to Δ, we would not reach a logical contradiction; so it would be wrong to say the statement was false. We reserve the word 'false' for when the addition of the statement generates a contradiction.

Definition 3.16 (Falsehood in a Frame) If $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$ is a frame, and $A \subset \mathcal{E}$, and $x \in \mathcal{E}$, then we say x is *false in* $\mathcal{F}$ *with respect to* A if and only if A and x have the following properties:
(1) x is not a paradox in $\mathcal{F}$.
(2) A is not contradictory in $\mathcal{F}$.
(3) $A \cup \{x\}$ is contradictory in $\mathcal{F}$.

The statement, *'There exists an x such that $x \neq x$,'* is not contradictory in $\mathcal{F}_L$ to itself. Only when combined with the Axiom of Equality (Axiom **??**) does it generate a contradiction. Thus, *'There exists an x such that $x \neq x$,'* is false in $\mathcal{F}_L$ with respect to Δ.

Consider a frame, $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$, any basis set $A \subset \mathcal{E}$, and any $x \in \mathcal{E}$. If A is contradictory in $\mathcal{F}$, or if x is a paradox in $\mathcal{F}$, then x can be neither true nor false in $\mathcal{F}$ with respect to A. So suppose A is not contradictory in $\mathcal{F}$, and x is not a paradox in $\mathcal{F}$; then x can still be neither true nor false in $\mathcal{F}$ with respect to A, as in the example of the statement about apples in our logical system. On the other hand, if we put

[3] A phrase coined by Wittgenstein in his book 'Philosophical Investigations'.

no restrictions upon $\mathcal{F}$, then it is possible for x to be both true *and* false in $\mathcal{F}$ with respect to A. However, if $\mathcal{F}$ is thorough and proper, then x cannot be both true and false in $\mathcal{F}$ with respect to A. I invite you to prove this for yourself.

Now we can replace the phrase 'I believe the following...' with 'The following is true in my mind with respect to my experience of the world...' If someone else does not believe what we believe, we should not be surprised. Truth in a frame depends upon the nature of the frame, and upon the basis set. Not all people reason in the same way, and each person has different experiences of the world. Convincing someone of something can be more arduous than simply reciting one's line of reasoning. People will not be swayed by our reasoning if the way they think is very different from ours. Even if they think as we do, their experience of the world may not include all the things which have led us to our beliefs. If we tell someone they are wrong merely because their conclusions are different from ours, they need pay no more attention to us than to say: 'In the mind of this person, and with respect to their experiences of the world, it is true that I am wrong'.

Presumably, we think in the way that seems best to us, so all of us are tempted to think that truth in our own minds is the ultimate truth. If we succumb to this temptation, then we will separate people into two types. There are the wise ones, who agree with us, and there are the mislead ones, who do not agree with us. When a person propounds a theory which is grossly at variance what we believe, we might be tempted to heap scorn upon them. Perhaps they will respond by heaping scorn upon us in return. How can such an argument be resolved?

One way is for the participants to agree to conduct the argument in a well-defined communicative frame. The Frame of Restricted Mathematical English and Logic is an example of such a frame. However, before we could conduct the argument, we would have to agree about our axioms. This would be revealing in itself. Then we could find out whose beliefs were true in the agreed frame with respect to the agreed axioms. If it turns out that in the agreed frame our beliefs are *not* consequences of our axioms, then we must admit defeat. We take this risk when we agree to argue in a particular communicative frame. The obvious disadvantage of this method is that we are not guaranteed to find ourselves correct. Our minds may be following lines of reasoning which have no parallel in the communicative frame. In order to be sure of proving our point, we might have to choose a frame in which we know the point can be proved from axioms our opponents would be prepared to accept. If we believe in dragons, we may insist upon the dragon frame. In $\mathcal{F}_{\text{dragon}}$, every set of axioms leads to the conclusion, *'Dragons once roamed the world.'* However, the people with whom we are arguing may not agree to such a frame. And they are all likely to have their own favourite frames, well suited to proving their own opinions.

Another way to conduct an argument is for each participant to describe their reasoning with the help of a frame and a set of axioms. Instead of reaching agreement about the axioms, each participant merely lists his axioms in a language everyone present understands. Then each participant describes a frame whose domain is the shared language, whose legal subset represents his relevent beliefs on impossibility, and whose consequence mapping represents his intuition. If this is done clearly and honestly by everyone, then the exact differences of opinion will be made plain, and the participants shall at least understand one another — even if they still do not agree. Perhaps in trying to express ourselves clearly, we might discover weaknesses in our arguments. Or perhaps we might decide that some of our axioms were incorrect when we had heard what others had to say. But we shall always be able to stick with our opinions if we so choose. There is no need for us ever to swallow our pride and admit defeat.

Never admitting defeat is all very well if one is never mistaken, or if one can be satisfied by self-confidence alone, but if we are trying to earn pleasures from a world whose laws are not affected by our beliefs, then we will not be satisfied by self-confidence alone. We will be satisfied only by success in our efforts to obtain from the world around us what we want of it. For example, our greatest ambition might be to attract a wonderful spouse. Then it would not matter how worthy of a wonderful spouse we believed ourselves to be, it would matter to us only whether or not we attracted one. If we are failing in this ambition, we shall have to think about our experiences and decide what to do next. When we think, we use a system of reasoning. This system can be represented by a frame, $\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mathcal{T})$. Our experiences are represented by some $A \subset \mathcal{E}$. Our decisions are consequences in $\mathcal{F}$ of A. By choosing $\mathcal{F}$ to guide us, we trust that the consequences in $\mathcal{F}$ of A correspond to the consequences in the world around us of the experiences represented by A. If the way consequences arise in $\mathcal{F}$ out of our experiences is at variance with the way in which events follow one another in our environment, then we shall find that our decisions are not successful. In order to get more of what we want out of a world which does not bow to our beliefs, we should try to use a frame in which consequences arise out of experiences according to the same patterns by which events follow one another in the world around us. Our own minds may not have this property. We may find our intuition does a poor job of making predictions. Perhaps our intuition is confounded by pride, or by fear, or by plain ignorance. We could then decide to mistrust our intuition, and choose to trust another frame instead — perhaps $\mathcal{F}_L$, or $\mathcal{F}_{\text{dragon}}$. If we chose $\mathcal{F}_L$, then we would represent our experiences as a set of statements in E_L, and we would use their consequences in $\mathcal{F}_L$ to guide us. Of course, if $\mathcal{F}_L$ produced bad predictions, then we would have to reconsider our choice of frame.

3.12 The World Within a Frame

In this section, I show how it is possible to construct a frame representing the workings of the world around us. We will use this frame in the next chapter.

You may not believe that anything outside your own mind exists at all. You could choose to believe that life is but a dream. If you did, then you would have little use for a frame representing the workings of the so-called 'world around us', because you would not believe that any such thing existed. But you would have a use for a frame representing the behaviour of your own dreaming mind, and the following exercise will give you some ideas about how you might construct such a frame.

There are many ways of looking at life: it could be an illusion, a game, or a trial of the spirit by an omnipotent god. The view I have chosen to represent with a frame in this section is one I shall call 'externalism'.[4] Externalists assume the existence of a very large thing containing not only our minds, but a great many other things as well. Externalists have a name for this very large thing whose existence they assume. They call it the 'universe'.

Externalism defines a type of thing called an 'event' in the same way as mathematics defines a type of thing called a 'set'. The universe is said to be made up of events. The universe itself is one big event, and any part of it is also an event, but *not every event is part of the universe*. If an event is part of the universe, then it is said to 'occur', and we call it a 'fact'. Otherwise it 'does not occur', and it is not a 'fact'. If an event occurs, then it may stimulate our senses, in which case we say we 'experience' the event.

According to externalists, our experience of an event is distinct from the event itself. This distinction is a result of the way externalism defines the words 'event' and 'experience'. According to these definitions, there is more to an event than our experience of it. For example, although we may agree that the sky is blue, we are unlikely to agree upon the best way to imagine the magnetic and electrostatic fields which science tells us are the constituents of light. This is similar to the question of how we should imagine numbers. Each of us has his or her own idea of what a number is like. We only agree upon their mathematical properties. When it comes to defining events, we are faced with the same difficulties as when we try to define numbers. We do not avoid the problem of describing numbers by saying they are sets. We merely turn the problem into one of defining sets. Only the mathematical properties of sets are defined rigorously. The other properties are loosely defined. The image of them as 'fantasy boxes' is far from strict. The boxes I see are perfect cubes of a metallic grey. Yours may be multicoloured for all I know.

[4] And I shall refer to upholders of externalism as 'externalists'

The only properties of an event we are likely to agree upon are those deciding what it is like to experience the event. We agree that the sky is blue, but we do not agree about the appearance of electrostatic fields. The problems we encounter when describing events are similar to those we encounter when describing abstract objects; but it would go against the common use of the word 'abstract' if I were to say events in the universe were abstract. We shall regard events as being in a class of their own — a class which is not abstract.

Let us define $\mathcal{E}_U$ to be the set of all events. The universe is made up of events, so we can represent the universe with a set $U \subset \mathcal{E}_U$, where U is the set of all events which occur in the entire history of the universe. Thus U is the set of all facts. As mentioned earlier, the universe itself is an event, so it is a member of U. The set U is only a *representation* of the universe. It is not the universe itself. We have $U \notin U$.

Although U represents all the facts, it does not represent the laws by which the universe might be governed. I shall represent these laws with the legal subset and consequence mapping of a frame $\mathcal{F}_U = (\mathcal{E}_U, \mathcal{L}_U, \mathcal{T}_U)$. The way I do this is similar to the way I represented the rules of logic with $\mathcal{L}_L$ and $\mathcal{T}_L$.

We say an event is 'inherently impossible' if the event on its own violates the laws of our universe. An example of an event which might be inherently impossible is the event of the speed of a particle having more than one value at the same time. In the construction of $\mathcal{F}_L$ I said $\mathcal{L}_L$ was the set of all statements which are *not* of the form '$A\overline{A}$'. Here I shall say $\mathcal{L}_U$ is the set of all events which are *not* inherently impossible.

We say an event is 'circumstantially impossible' if it is not inherently impossible but all the same cannot occur in our universe because other events have taken place which preclude its occurance. The events which have already occured are the 'circumstances', hence the phrase 'circumstantially impossible'. If a particle is not moving, then the event of its moving is circumstantially impossible, since we would then have a particle with two values of speed.

In logic, there are rules which identify new theorems on the strength of existing theorems. A basis set has theorems even when it is logically contradictory. We can apply the rules of logic even to statements of the form '$A\overline{A}$'. Just as we take the rules of logic and allow ourselves to apply them to basis sets which are logically contradictory, we shall take the laws of consequence which operate in our universe and allow ourselves to imagine them operating upon impossible collections of events.

Impossible collections of events are, by definition, those which cannot arise in our universe. We merely imagine them for convenience. We imagine their consequences for convenience as well. First we imagine an impossible collection of events, then we imagine other events occuring as a result of our impossible collection. These other events occur because of our universe's laws of consequence. This imagined process of cause and effect leads eventually to the occurance of an event which is inherently impossible. When this happens, we know our starting collection of events must indeed be impossible.

Consider an example: suppose we have the following collection of events:

(1) A cricket ball is alone in space.
(2) No forces are acting upon the cricket ball.
(3) The cricket ball is accelerating.

Newton's First Law states that a body will not accelerate unless acted upon by a force. If we suppose that Newton's First Law is one of the laws of consequence in our universe, then it would be a consequence of (1) and (2) that the ball was *not* accelerating. But we have event (3) in our collection as well. When we add this to the event of the ball not accelerating, we get the event of the ball both accelerating and *not* accelerating. Now let us suppose that this event of both accelerating and not accelerating is inherently impossible in our universe. Then we conclude that the collection of events (1) to (3) is impossible, for it has an inherently impossible consequence. No possible collection of events could ever have an inherently impossible consequence. Note that event (3) is circumstantially impossible whenever events (1) and (2) are occuring, and event (2) is circumstantially impossible whenever events (1) and (3) are occuring.

We define $\mathcal{T}_U$ in the following way. Let A be any set of events. The universe's laws of consequence, when applied to the events of A, will dictate that certain other events must occur as a consequence of the events in A. Not least of these consequences are the events in A themselves, just as the first logical consequences of a basis set are the basis statements themselves. Given any $A \subset \mathcal{E}_U$, we define $\mathcal{T}_U$ to be such that $\mathcal{T}_U(A)$ is the set of all events which *must* occur as a result of the events in A. Because of our imagined extention of the laws so that they can be applied to impossible collections of events, we see that we have allowed $\mathcal{T}_U(A)$ even when A is an impossible set of events.

Now we can say 'consequence in $\mathcal{F}_U$' instead of 'caused by the universe's laws of consequence'. If A is an impossible set of events, then it has a consequence in $\mathcal{F}_U$ which is not a member of $\mathcal{L}_U$. Thus, A is impossible if and only if it is contradictory in $\mathcal{F}_U$ (see Definition **??**).

Every event has at least one consequence: itself. If A is a set of events, then every member of A is a consequence in $\mathcal{F}_U$ of A. So for every $A \subset \mathcal{E}_U$ we have $A \subset \mathcal{T}_U$. This means $\mathcal{F}_U$ has property (1) of proper frames (see Definition **??**).

The laws of consequence of which I have been speaking are those laws which dictate what events *must* occur as a result of other events. They do not tell us what *might* occur. If A is a set of events, then $\mathcal{T}_U(A)$ is the set of all events which *must* occur as a result of the members of A. That is to say, if every event in A occurs, then nothing can prevent any event in $\mathcal{T}_U(A)$ from occuring. If I drop an apple, the laws of consequence do not dictate that the apple falls to the earth. That is merely something that *might* happen. Many other things could happen. The apple could be swept up by an eagle, or it could fall on someone's head. Only the inevitable consequeces of A are to be found in $\mathcal{T}_U(A)$. This means that if we add events to A, so as to make a larger set of events, B, then every consequence in $\mathcal{F}_U$ of A will also be a consequence in $\mathcal{F}_U$ of B. For every A and B such that $A \subset B \subset \mathcal{E}_U$, we have $\mathcal{T}_U(A) \subset \mathcal{T}_U(B)$. This means $\mathcal{F}_U$ has property (2) of proper frames.

If the rules of consequence are applied to an empty collection of events (a collection containing no events at all), then the rules shall have nothing to work upon, so there will be no consequences. That is, the inevitable consequences of nothing are also nothing. Thus $\mathcal{T}_U(\emptyset) = \emptyset$. Theorem **??** implies $\mathcal{T}_U(\emptyset) = \emptyset \subset \mathcal{L}_U$, so $\emptyset$ is not contradictory in $\mathcal{F}_U$. This means $\mathcal{F}_U$ has property (3) of proper frames, as well as properties (1) and (2). So $\mathcal{F}_U$ is a proper frame.

The proof that $\mathcal{F}_U$ is proper cannot be declared as a formal numbered theorem, because $\mathcal{F}_U$ does not exist with respect to Δ. We shall not add to Δ any declaration of the existence of $\mathcal{F}_U$, for to do so would be to declare in Δ that externalist universe existed. We do not want to alienate those readers who are not externalists.

Some people believe the universe obeys laws which are scientific in nature. We call these people 'scientific externalists'. In scientific externalism, it is possible to show that $\mathcal{F}_U$ is thorough as well as proper. You may like to think about this yourself.

I shall now prove that an event is inherently impossible if and only if it is a paradox in $\mathcal{F}_U$. For the first half of the proof, let x be any inherently impossible event. Since $\mathcal{F}_U$ is proper, x is a consequence in $\mathcal{F}_U$ of itself, so x has at least one inherently impossible consequence in $\mathcal{F}_U$, so x is a paradox in $\mathcal{F}_U$. For the second half of the proof, let x be a paradox in $\mathcal{F}_U$. This means $\mathcal{T}_U(\{x\})$ contains an inherently impossible event (one which is not a member of $\mathcal{L}_U$), so x on its own violates the laws of the universe, so x is also an inherently impossible event.

The following three statements are equivalent:
(1) x is inherently impossible.
(2) x is a paradox in $\mathcal{F}_U$.
(3) $x \in \mathcal{E}_U$ and $x \notin \mathcal{L}_U$.
Number (3) is obvious, because $\mathcal{L}_U$ is the set of all events which are not inherently impossible. No event in $\mathcal{L}_U$ is contradictory in $\mathcal{F}_U$ to itself. In this way $\mathcal{F}_U$ differs from $\mathcal{F}_L$. In $\mathcal{F}_L$, statements like, '*It is raining and it is not raining,*' or, '*There exists a statement, A, which is the the statement, "A is not a theorem"*

,' are paradoxes in $\mathcal{F}_L$ which are members of $\mathcal{L}_L$.[5]

If A is not contradictory in $\mathcal{F}_U$, then some events can be circumstantially impossible with respect to A. An event, x, is circumstantially impossible if and only if it is on the one hand not inherently impossible, but on the other hand forms an impossible set of events when added to A. Thus x is circumstantially impossible if it is not a paradox in $\mathcal{F}_U$ and $\{x\} \cup A$ is contradictory in $\mathcal{F}_U$. In other words, x is circumstantially impossible with respect to A if and only if x is false in $\mathcal{F}_U$ with respect to A.

If U is the set of all events which occur in the entire history of the universe, then we see that every consequence of every one of these events must also be a member of U. Thus U is such that $\mathcal{T}_U(U) = U$. Now let B be a set of events representing the motion of every particle in the universe at the some particular instant in time. If you believe that the movements of all the particles in the universe at any instant in time dictate everything that comes after, and specify everything that must have come before, then you believe that $U = T_U(B)$.

Given any two different events, we can always find a third event which is a consequence of one event but not a consequence of the other. No two unequal events are equivalent in $\mathcal{F}_U$ to one another (see Definition **??**). This is an important property of $\mathcal{F}_U$. It is a property $\mathcal{F}_L$ does not posess. Every $x \in \mathcal{E}_L$ has many statements which are equivalent to it in $\mathcal{F}_L$. For example, *'It is daytime,'* is equivalent in $\mathcal{F}_L$ to, *'It is daytime and it is daytime'*, and so on.[6]

If you believe life is a dream, or have some other disagreement with the view described by $\mathcal{F}_U$, then I invite you to construct a frame representing your own view. When you read $\mathcal{F}_U$ in the text of the following chapters, you can substitute your own frame and judge my arguments on your own terms.

3.13 Final Remark

Our minds, our systems of reasoning, and the world around us, may be represented by frames. When it comes to arguing with people, we can choose a communictive frame to suit our conclusions, or we can choose a communicative frame for some other reason. We may want to choose a frame which acts as an effective model of the world around us. In the next chapter we shall talk of 'symbolisms' and 'strict symbolisms'. With these tools we shall be able consider what it is for a frame to act as an 'an effective model of the world around us'.

[5] Neither statement has the exact form '$A\overline{A}$'.

[6] This gives us a trivial way of generating an infinite number of equivalent statements in $\mathcal{F}_L$.

Chapter 4

Symbolisms

As I write this book, I am translating ideas into written statements. There is a relationship between my mind and the statements on this page. In the last chapter I showed that my mind could be represented by a frame $\mathcal{F}_A = (\mathcal{E}_A, \mathcal{L}_A, \mathcal{T}_A)$. I also showed that the system of reasoning I am using to describe philosophical tools can be represented by a frame $\mathcal{F}_L = (\mathcal{E}_L, \mathcal{L}_L, \mathcal{T}_L)$. In this chapter, I shall show that the way I translate ideas into written statements can be represented by a relationship between $\mathcal{F}_A$ and $\mathcal{F}_L$. Such a relationship is called a 'symbolism'.

4.1 Introducing Symbolisms

To prepare us for the definition of symbolisms, let us consider in more detail the way I use written statements to express my ideas. My ideas are members of $\mathcal{E}_A$. The set of all Restricted Mathematical English statements is called $\mathcal{E}_L$. The statements I write here are members of $\mathcal{E}_L$. There are some concepts which I believe myself to be incapable of writing down, but my ideas about philosophy are not among them. Let us define $\mathcal{D}_{AL} \subset \mathcal{E}_A$ to be the set of all thoughts in $\mathcal{E}_A$ which are thoughts on subjects related in this book.[1]. For each idea in $\mathcal{D}_{AL}$, there is at least one statement in $\mathcal{E}_L$ which I think expresses the idea. Given an idea in $\mathcal{D}_{AL}$, there are always many statements which I think express the idea equally well. For example, consider the idea of the Axiom of Equality. As far as I am concerned, each of the following statements is an equally valid expression of this idea:
(1) Everything is equal to itself.
(2) For every x we have $x = x$.
(3) There exists no x such that $x \neq x$.

For every idea in $\mathcal{D}_{AL}$, there is a collection of statements which are each, to my thinking, equally valid expressions of the idea. The association in my mind between ideas and collections of statements may be represented with a mapping. The domain of this mapping is $\mathcal{D}_{AL}$. Its target is the set of all subsets of $\mathcal{E}_L$ — otherwise known as $\wp(\mathcal{E}_L)$. The mapping pairs each idea with a set of statements. The set of statements with which an idea is paired is the set of all statements which I think express the idea. We do not include in the set any statements which express ideas in addition to the particular idea with which we are concerned. If x is the idea of the Axiom of Equality, then the set of statements with which it is paired contains statements (1) to (3) above, but *not* the statement, *'Everything is equal to itself, and the weight of the largest living brontasaurus is 500 tons.'* This statement is excluded because it expresses ideas in addition to that of the Axiom of Equality.

Let $\mathcal{S}_{AL}$ denote the mapping described above. The mapping $\mathcal{S}_{AL}$ represents the way I use statements in $\mathcal{E}_L$ to express ideas in $\mathcal{D}_{AL}$. Every $x \in \mathcal{D}_{AL}$ is paired with a set of statements $\mathcal{S}_{AL}(x) \subset \mathcal{E}_L$ (or, equivalently,

[1] The subscript 'AL' will make sense to you later

$\mathcal{S}_{AL}(x) \in \wp(\mathcal{E}_L)$). The mapping $\mathcal{S}_{AL}$ is an example of a symbolism. We defined $\mathcal{S}_{AL}$ without referring to the frames $\mathcal{F}_A$ and $\mathcal{F}_L$. We referred only to their domains, $\mathcal{E}_A$ and $\mathcal{E}_L$. All the same, because $\mathcal{E}_A$ is the domain of $\mathcal{F}_A$, and $\mathcal{E}_L$ is the domain of $\mathcal{F}_L$, we say $\mathcal{S}_{AL}$ is a 'symbolism from $\mathcal{F}_A$ to $\mathcal{F}_L$'. Thus, we arrive at the following definition:

Definition 4.1 (Symbolisms) If $\mathcal{F}_1 = (\mathcal{E}_1, \mathcal{L}_1, \mathcal{T}_1)$ and $\mathcal{F}_2 = (\mathcal{E}_2, \mathcal{L}_2, \mathcal{T}_2)$ are frames, then a *symbolism from* $\mathcal{F}_1$ *to* $\mathcal{F}_2$ is any mapping from a subset of $\mathcal{E}_1$ to the power set of $\mathcal{E}_2$.

We also define the 'symbol set' of a symbolism:

Definition 4.2 (Symbol Set of a Symbolism) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are frames, and $\mathcal{S}_{12}$ is a symbolism from $\mathcal{F}_1$ to $\mathcal{F}_2$ whose domain is $\mathcal{D}_{12}$, then the *symbol set* of $\mathcal{S}_{12}$ is the set $\{x \mid x \in \mathcal{S}_{12}(y) \text{ for some } y \in \mathcal{D}_{12}\}$.

The symbol set of $\mathcal{S}_{AL}$ is the set of all statements which could be called upon to express ideas in $\mathcal{D}_{AL}$. Notice that we do not have to define what the 'domain' of a symbolism is, since this has already been done in Definition **??**.

Let us define another symbolism, $\mathcal{S}_{LR}$, to represent the way *your* mind interprets written statements. We let the domain of $\mathcal{S}_{LR}$ be the same as the symbol set of $\mathcal{S}_{AL}$. In this way, $\mathcal{S}_{LR}$ includes your interpretation of all statements I might use in this book.

Let $\mathcal{D}_{LR}$ denote the domain of $\mathcal{S}_{LR}$. For every statement in $\mathcal{D}_{LR}$, the mapping $\mathcal{S}_{LR}$ pairs the statement with the set of ideas you think are the possible meanings of the statement. In the last chapter we represented the reader's mind with a frame $\mathcal{F}_R = (\mathcal{E}_R, \mathcal{L}_R, \mathcal{T}_R)$. Thus $\mathcal{S}_{LR}$ is a symbolism from $\mathcal{F}_L$ to $\mathcal{F}_R$.

The symbol set of $\mathcal{S}_{LR}$ is the set of all interpretations you could make of statements in $\mathcal{D}_{RL}$. The set $\mathcal{D}_{RL}$ is equal to the symbol set of $\mathcal{S}_{AL}$, so it is the set of all statements I might call upon to express philosophical ideas. Consequently, the symbol set of $\mathcal{S}_{LR}$ can be regarded as the set of all possible understandings you might have of my philosophical statements.

We now have two symbolisms representing the way ideas are communicated from my mind to yours.[2] We can combine these into one symbolism. Let $\mathcal{S}_{AR}$ be a symbolism from $\mathcal{F}_A$ to $\mathcal{F}_R$ such that for all $x \in \mathcal{D}_{AL}$ we have $\mathcal{S}_{AR}(x) = \{y \mid y \in \mathcal{S}_{LR}(s) \text{ for some } s \in \mathcal{S}_{AL}(x)\}$. That is, $\mathcal{S}_{AR}(x)$ is the set of all interpretations you could make of all the statements I might use to express x. We would like there to be only one way for you to interpret what I have said. My communication would then be unambiguous. Of course, there is more to effective communication than unambiguity. We would also like $y \in \mathcal{S}_{AR}(x)$ to be similar to x.

4.2 Similar Ideas

What is it for an idea in your mind to be 'similar' to one in mine? In asking such a question, we presuppose the existence of minds outside our own. Those who believe that life is but a dream would not agree with this supposition. Consequently, they would consider the question to be meaningless.

But let us define what it is for ideas to be similar within the context of externalism.[3] In the last chapter, I represented the externalist universe with the frame $\mathcal{F}_U$, where $\mathcal{F}_U = (\mathcal{E}_U, \mathcal{L}_U, \mathcal{T}_U)$.

First we must recognise that although $\mathcal{F}_U$ represents the externalist universe and its governing laws, it does not represent all assumptions externalism makes about the nature of 'events'. We recall that 'event'

[2]To represent the way you use written statements to communicate ideas, we could define $\mathcal{S}_{RL}$ in the same way we defined $\mathcal{S}_{AL}$. To represent the way I interpret written statements, we could define $\mathcal{S}_{LA}$ as we defined $\mathcal{S}_{LR}$. Because the way we interpret statements is closely related to the way we use them to express ourselves, we are likely to find that $\mathcal{S}_{AL}$ is simply the reverse of $\mathcal{S}_{LA}$, and $\mathcal{S}_{RL}$ is the reverse of $\mathcal{S}_{LR}$.

[3]Externalism, you will recall, is the belief that there is a universe extending beyond our minds.

is a word defined by externalists to help describe their view of the universe. Mathematicians did a similar thing by defining the word 'set' to help describe the properties of numbers. To be consistent with our definition of 'sets', let us also use postulates to define 'events'. We shall need two such postulates. The first of these is that every part of the universe is an event (this does *not* mean that every event is a part of the universe). The second postulate is that every thought of ours is a mental image of a unique event. That is, to every thought, x, there corresponds one and only one event, which we shall refer to as the 'event thought of by x'. We shall use symbolisms to represent this correspondance.

Let $\mathcal{S}_{AU}$ be a symbolism from $\mathcal{F}_A$ to $\mathcal{F}_U$. The domain of $\mathcal{S}_{AU}$ is the whole of $\mathcal{E}_A$. For any $x \in \mathcal{E}_A$ we have $\mathcal{S}_{AU}(x) = \{\text{the event thought of by } x\}$. Likewise we have $\mathcal{S}_{RU}$ from $\mathcal{F}_R$ to $\mathcal{F}_U$ such that for any $z \in \mathcal{E}_R$ we have $\mathcal{S}_{RU}(z) = \{\text{the event thought of by } z\}$. Although we have said that for every thought there is a unique event, we have not said that for every event there is a unique thought. Given an event, there may be no thought corresponding to it, or there may be many thoughts corresponding to the same event. If we distinguished the thought of the Axiom of Equality on Tuesday from the thought of it on Monday, then we would have two thoughts corresponding to the same event.

If x and y are two thoughts, either thoughts in the same mind or thoughts in different minds, then we say x and y are 'similar' if and only if they are thoughts of the same event. For you and I, this means $x \in \mathcal{E}_A$ is similar to $y \in \mathcal{E}_R$ if and only if $\mathcal{S}_{AU}(x) = \mathcal{S}_{RU}(y)$.

4.3 Sentient Beings

We might also ask, 'What is a thought, and what is not a thought?' To introduce some of the problems that arise in our efforts to define thought, and what it is to be capable of thought, let us enjoy a little rambling commentary on the matter of 'sentience':

'One might believe that trees are sentient creatures, or perhaps that the earth is a slow-thinking creature married to the sun. The arguments in favour of these beliefs may be weak, but careful thought will reveal that they are no weaker than the arguments in favour of regarding our fellow humans as sentient. Arguments are lent no more strength by the fact that we belive the assumption they are trying to justify. After all, who is to say other humans are not merely automatons responding by remote control to the commands of some central organising machine which, though the rules by which it runs are simple, gives the impression of complex intelligence, because of the sheer number of components in its calculations? How do we define sentience so that it will be a property independent of our perspective? We might say a sentient being is anything which can respond to stimuli and learn from its experiences. But a rock responds to the blow of a chisel by being chipped, and will never allow itself to be chipped the same way again, for the spot that was chipped has fallen away. Does that constitute a learned response to stimuli?'

The word 'meal' is used to indicate something we think should be subjected to chewing and swallowing. Although we agree about what it is to treat something as a meal, we may not agree about what things one should treat as meals. A Chinese chef is likely to regard a dog as a meal, while an English spinster would regard it as a pet. I regard trees as a mindless form of life, while King George III is rumored to have treated them as close friends.

I regard you, my reader, as a 'sentient being'. I choose to describe the processes governing your behaviour by defining symbolisms to and from a frame representing what I call your 'mind'. Perhaps I am mistaken. Perhaps I should stop writing a book for you and treat you as I would a tree. Or perhaps, although you are quite mindless, my way of treating you is by coincidence still appropriate.

To say something is a 'sentient being' is to say we believe the best way to understand the thing's behaviour is to think of it as being guided by a mind which imagines and experiences events. I do not regard trees as sentient beings, because I do not think it is necessary to do so.

4.4 Accurate Communication

Suppose I try to communicate an idea, x, and my efforts conjure an idea, y, in your mind. Ideally, we would have x similar to y. My communication could then be said to be 'accurate', for all the details of x would have been communicated without embellishment. But we cannot expect our communication to be accurate every time, so we shall define some words to describe partially effective communication. We shall define these words within the context of externalism.

Suppose I use a written statement $p \in \mathcal{E}_L$ to express an idea $x \in \mathcal{D}_A$. What you see as the possible meanings of p are the members of $\mathcal{S}_{LR}(p)$. Let x' be the event thought of by x, and let Y' be the set of events thought of by the members of $\mathcal{S}_{LR}(p)$. We say the communication is 'unembellished' if and only if every event in Y' is a consequence in $\mathcal{F}_U$ of x'. We say the communication is 'complete' if and only if x' is a consequence in $\mathcal{F}_U$ of every event in Y'. If my communication is both unembellished and complete, then it must be accurate, since no two unequal events are equivalent in $\mathcal{F}_U$ (p **??**).

Communication of an idea is unembellished if some or all of the details of the idea are communicated without embellishment. The way we decide what counts as embellishment is by referring to $\mathcal{F}_U$, the frame representing the externalist universe. In the externalist model, the idea I try to communicate always corresponds to an event. Using the symbols defined in the preceding paragraph, suppose $\mathcal{S}_{LR}(p)$ contains only one member, y. This means we have assumed my communication is unambiguous, for y is the only meaning you feel can be attached to p. We let y' denote the event thought of by y. If y' has a detail which is not a consequence in $\mathcal{F}_U$ of x', then x has been embellished in the process of communication. Communication of x is complete if all of the details of x are communicated — with or without embellishment. The way we decide whether y contains all details of x is once more by referring to $\mathcal{F}_U$. If x' is a consequence in $\mathcal{F}_U$ of y', then all details of x have been communicated. As an illustration, suppose x is the idea of a red ball flying alone through space, suppose p is the statement, '*A coloured ball flies through the void,*' and suppose $\mathcal{S}_{LR}(p)$ contains only one idea, y. If y is the thought of some ball of unspecified colour flying alone through space, then the communication of x is said to be unembellished, but not complete. The communication is unembellished because a ball of unspecified colour flying through space is a consequence in $\mathcal{F}_U$ of a red ball flying through space. The communication is not complete because a red ball flying through space is not a consequence in $\mathcal{F}_U$ of a ball of unspecified colour flying through space. On the other hand, if y is the thought of a red ball flying through space with a star in the background, then the communication is not unembellished, but it is complete. It is complete because a red ball flying through space is a consequence in $\mathcal{F}_U$ of a red ball flying through space with a star in the background. The communication is not unembellished, because the information about the star is an embellishment of the original idea.

You may have noticed that in the last paragraph I expected you to agree with me when I made statements about consequences in $\mathcal{F}_U$. The example was simple, so you probably did agree. But if it were more complicated — perhaps containing a few ethical theories — then I could not expect you to agree with me. The externalist universe is the foundation of our definitions of unembellished and complete communication. We can agree upon the meanings of these definitions only so far as we can agree upon the nature of the externalist universe.

We have defined what it is for communication to be accurate. Now we address the problem of detecting inaccuracy in our communication. This problem is significant because our only connection with other people's minds is through language.

Suppose I describe an idea $x \in \mathcal{E}_A$, and you interpret me as meaning $y \in \mathcal{E}_R$. To test the accuracy of my communication, the best I can do is ask you to describe y to me. This requires a second effort of communication which might introduce inaccuracies of its own. Let $z \in \mathcal{E}_A$ be the meaning I attach to your description of y (assuming there is only one such meaning). If z is not similar to x, then we know at least one of our efforts to communicate was not accurate. By examining the difference between z and x, we might discover the source of the inaccuracy. We shall call this test for inaccuracy the 'reply test'.

Even if we apply the reply test, and z and x turn out to be similar, we still cannot be certain that y is similar to x. For example, to describe y to me, you could repeat my own statement, p. I would be inclined to interpret p as meaning x, since I used p to express x in the first place. But you might have had no understanding of p at all.[4] For the reply test to be effective, you should describe y to me with some statement other than p. If you are unable to find a statement other than p to express what you see as the meaning of p, then we are less able to detect inaccuracy in our communication.

If we have agreed to conduct our discussion in $\mathcal{F}_L$, then choosing an alternative to p should not be difficult. There should be many statements which are equivalent in $\mathcal{F}_L$ to p, but which have significantly different wording. For example, if I said to you, 'Everything is equal to itself,' then you could reply, 'For every x we have x equal to x,' or even, 'Nothing is unequal to itself.' Of course, if you did not think these statements were equivalent expressions of the same idea, then $\mathcal{F}_L$ is not a suitable frame for you to express your reasoning.

For the reply test to be effective, we must insist upon alternative wording in the reply. We shall call this stricter test the 'Equivalent Alternative Reply' test (EAR test). If our communication is to be accurate, then we must take pains over our use of language. We must be diligent in our application of the EAR test. If we consider inaccuracy to be such a bad thing that nothing is worth saying if it is going to be interpreted inaccurately, then we should refrain from trying to express ideas for which we cannot find equivalent alternative expressions, for then we cannot apply the EAR test to our communication. When we cannot apply the EAR test, our efforts to communicate can be grossly inaccurate without our ever knowing it (as I hope to demonstrate in the next paragraph). Of course, even if a communication passes the EAR test, it is still not *guaranteed* to be accurate. To be more certain that there is no inaccuracy, we can apply the EAR test to the expression of many ideas related to the one we originally wanted to communicate. Such scrutiny would amount to a more thorough search for differences in our use of language. If we still cannot detect any inaccuracy, then we could argue that what inaccuracy exists is not worth worrying about.

We should note that accuracy may not be such a vital property of communication. Many of our greatest philosophers have attached little importance to it. For example, here is an aphorism for which everyone has a different interpretation, but which is none the less held by many to be laden with wisdom:

'Life is a fullness you cannot abandon.' [5]

How very true that is. Straight to the point. But what point? Can we rephrase this point into another equivalent statement? Try it, and see if anyone agrees with you. If it were that easy, then the quote would not be so vital to the expression of the idea. The original sentence would have an infinity of equivalent brothers, and people would use which ever one sprang to mind. The idea would not be tied to the quotation — just as Newton's laws are not tied to any quotation of his. Newton's laws may be expressed without difficulty in many equivalent ways.

Here is another aphorism. I use this one myself, but I admit that it is just as ambiguous as the aphorism I quoted above:

[4] I have been told the following story about the chess player Bobby Fischer. He called up the home of a Swedish friend. The friend was out, but his wife answered the telephone. She spoke no English, and Fischer spoke no Swedish. Still, she explained in a couple of Swedish sentences where her husband was. Then Fischer called up another friend who spoke Swedish, repeated the sentences verbatim, and had them translated.

[5] Sartre

'Seek freedom and become captive of your own desires, seek discipline and find your liberty.' [6]

4.5 The Operation of the Senses

We have defined symbolisms from our minds to the universe. Now we shall define symbolisms going the other way. These represent the operation of our senses within the context of externalism. They are not simply the reverse of the symbolisms from our minds to the universe. Many of the things we think about are events which we can never experience. We can think about an electron near an atom, but we can never see one with our own eyes. We can sympathise with another person's pain, but we cannot experience it. A man who has gone blind can imagine the colour blue, but he can no longer see it himself.

So let $\mathcal{D}_{UA}$ be the set of all events I can experience with my senses. We define $\mathcal{S}_{UA}$ to be a symbolism from $\mathcal{F}_U$ to $\mathcal{F}_A$. The domain of this symbolism is $\mathcal{D}_{UA}$, it pairs every event in $\mathcal{D}_{UA}$ with a set of ideas. For any event, $x \in \mathcal{D}_{UA}$, the set $\mathcal{S}_{UA}(x)$ contains the ideas which are the sensory impressions I would have of event x if it were to occur. In the same way, we let $\mathcal{D}_{UR}$ be the set of all events which you can experience with your senses. We define $\mathcal{S}_{UR}$ to be a symbolism from $\mathcal{F}_U$ to $\mathcal{F}_R$. Its domain is $\mathcal{D}_{UR}$. For an event, $x \in \mathcal{D}_{UR}$, the set $\mathcal{S}_{UR}(x)$ contains the ideas which are the sensory impressions you would have of event x if it were to occur. The set $\mathcal{D}_{UR}$ might be different from $\mathcal{D}_{UA}$. For example, you might be deaf, while I, at the time of writing, am not.

4.6 Representation in a Symbolism

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be frames, and let $\mathcal{S}_{12}$ be a symbolism from $\mathcal{F}_1$ to $\mathcal{F}_2$ whose domain is $\mathcal{D}_{12}$. Consider a set $X \subset \mathcal{D}_{12}$ which we would like to represent in $\mathcal{F}_2$ with the help of $\mathcal{S}_{12}$. For every $x \in X$ we choose one or more members of $\mathcal{S}_{12}(x)$ to represent x. Choosing more than one member of $\mathcal{S}_{12}(x)$ can be regarded as repetition. Once we have chosen a representation for each member of X, we bring them together to make a set $P \subset \mathcal{E}_2$. This set represents X. We call it a 'representation in $\mathcal{S}_{12}$ of X'.

Definition 4.3 (Representation in a Symbolism) If $\mathcal{F}_1 = (\mathcal{E}_1, \mathcal{L}_1, \mathcal{T}_1)$ and $\mathcal{F}_2 = (\mathcal{E}_2, \mathcal{L}_2, \mathcal{T}_2)$ are frames, $\mathcal{S}_{12}$ is a symbolism from $\mathcal{F}_1$ to $\mathcal{F}_2$ whose domain is $\mathcal{D}_{12}$, and $X \subset \mathcal{D}_{12}$, then we say $P \subset \mathcal{E}_2$ is a *representation in $\mathcal{S}_{12}$ of X* if and only if P has the following two properties:
(1) For every $x \in X$ such that $\mathcal{S}_{12}(x) \neq \emptyset$ there exists at least one $y \in \mathcal{S}_{12}(x)$ such that $y \in P$.
(2) For every $y \in P$ there exists $x \in X$ such that $y \in \mathcal{S}_{12}(x)$.

We see that for any $X \subset \mathcal{D}_{12}$ there may be many subsets of $\mathcal{E}_2$ which are representations in $\mathcal{S}_{12}$ of X. In the case of a set of ideas in my mind, $I \subset \mathcal{D}_{AL}$, there are an infinity of different statements in $\mathcal{E}_L$ to express any member of I, so there are an infinity of different representations in $\mathcal{S}_{AL}$ of I. However, for $I \in \mathcal{D}_{AU}$ there is only one representation in $\mathcal{S}_{AU}$ of I, because there is only one event which is thought of by each member of I.

So far, we have encountered several examples of communicative frames. The most memorable are $\mathcal{F}_{\text{dragon}}$ and $\mathcal{F}_L$. I am using $\mathcal{F}_L$ to communicate my ideas about frames and symbolisms. I think it is a useful communicative frame. However, $\mathcal{F}_{\text{dragon}}$ is a communicative frame about which we might have doubts. Can it be used to communicate our reasoning? This depends upon the way we think: if all we ever do is think about dragons, then $\mathcal{F}_{\text{dragon}}$ might serve us well. Otherwise its value is limited.

My purpose in the rest of this chapter is to define classes of symbolisms which will allow us to consider clearly whether or not a symbolism from one frame to another allows the properties of the first frame to be represented accurately within the second. I shall continue to refer to two unspecified frames $\mathcal{F}_1 =$

[6] Frank Herbert

$(\mathcal{E}_1, \mathcal{L}_1, \mathcal{T}_1)$ and $\mathcal{F}_2 = (\mathcal{E}_2, \mathcal{L}_2, \mathcal{T}_2)$, and a symbolism $\mathcal{S}_{12}$ from $\mathcal{F}_1$ to $\mathcal{F}_2$ whose domain is $\mathcal{D}_{12}$. Using unspecified frames means that the terminology I define can be applied to any symbolism. We could apply it to the use of $\mathcal{F}_{\text{dragon}}$ to represent the properties of $\mathcal{F}_R$, or to the use of $\mathcal{F}_L$ to represent $\mathcal{F}_U$, or even to the use of $\mathcal{F}_A$ to represent $\mathcal{F}_L$ (this last would be an effort on my part to think logically).

4.7 Classes of Symbolisms

As we mentioned above, for any $X \subset \mathcal{D}_{12}$ there may be many representations in $\mathcal{S}_{12}$ of X. A symbolism is 'proper' if all representations in $\mathcal{S}_{12}$ of X have the same consequences in $\mathcal{F}_2$. Then we can say that all representations in $\mathcal{S}_{12}$ of X are equivalent in $\mathcal{F}_2$ to eachother.

Definition 4.4 If $\mathcal{F}_1 = (\mathcal{E}_1, \mathcal{L}_1, \mathcal{T}_1)$ and $\mathcal{F}_2 = (\mathcal{E}_2, \mathcal{L}_2, \mathcal{T}_2)$ are frames, and $\mathcal{S}_{12}$ is a symbolism from $\mathcal{F}_1$ to $\mathcal{F}_2$ whose domain is $\mathcal{D}_{12}$, then we say $\mathcal{S}_{12}$ is *proper from $\mathcal{F}_1$ to $\mathcal{F}_2$* if and only if, for every $X \subset \mathcal{D}_{12}$, and for every P and Q which are representations in $\mathcal{S}_{12}$ of X, we have $\mathcal{T}_2(P) = \mathcal{T}_2(Q)$.

We have defined the phrase 'proper from $\mathcal{F}_1$ to $\mathcal{F}_2$' rather than the word 'proper' on its own, because $\mathcal{S}_{12}$ could also act as a symbolism between other frames besides $\mathcal{F}_1$ and $\mathcal{F}_2$. For example, $\mathcal{S}_{AL}$ is not only a symbolism between $\mathcal{F}_A$ and $\mathcal{F}_L$, but also a symbolism between $\mathcal{F}_A$ and $\mathcal{F}_{\text{dragon}}$. Whenever two frames have the same domain, any symbolism to one of them will also count as a symbolism to the other. The symbolism $\mathcal{S}_{AL}$ is proper from $\mathcal{F}_A$ to $\mathcal{F}_{\text{dragon}}$, and I would argue that it was proper from $\mathcal{F}_A$ to $\mathcal{F}_L$ as well. But it is easy to think up a frame, $\mathcal{F}$, whose domain is $\mathcal{E}_L$, for which $\mathcal{S}_{AL}$ is not proper from $\mathcal{F}_A$ to $\mathcal{F}$.

Let P be a representation of $X \subset \mathcal{D}_{12}$. If P has consequences in $\mathcal{F}_2$ which are not members of $\mathcal{S}_{12}$'s symbol set, then these consequences in $\mathcal{F}_2$ do not correspond through $\mathcal{S}_{12}$ to any member of $\mathcal{D}_{12}$. When P has consequences in $\mathcal{F}_2$ which are not members of $\mathcal{S}_{12}$'s symbol set, we say $\mathcal{S}_{12}$ is 'not thorough' from $\mathcal{F}_1$ to $\mathcal{F}_2$.

Definition 4.5 (Thorough Symbolisms) If $\mathcal{F}_1 = (\mathcal{E}_1, \mathcal{L}_1, \mathcal{T}_1)$ and $\mathcal{F}_2 = (\mathcal{E}_2, \mathcal{L}_2, \mathcal{T}_2)$ are frames, and $\mathcal{S}_{12}$ is a symbolism from $\mathcal{F}_1$ to $\mathcal{F}_2$ whose domain is $\mathcal{D}_{12}$ and whose symbol set is $\mathcal{G}_{12}$, then we say $\mathcal{S}_{12}$ is *thorough from $\mathcal{F}_1$ to $\mathcal{F}_2$* if and only if, for every $X \subset \mathcal{D}_{12}$, and for every P a representation in $\mathcal{S}_{12}$ of X such that P is not contradictory in $\mathcal{F}_2$, we have $\mathcal{T}_2(P) \subset \mathcal{G}_{12}$.

Using the symbols defined in Definition **??**, if a consequence in $\mathcal{F}_2$ of P is not a member of $\mathcal{G}_{12}$, and P is not contradictory in $\mathcal{F}_2$, then $\mathcal{S}_{12}$ is not thorough from $\mathcal{F}_1$ to $\mathcal{F}_2$. Of course, $\mathcal{S}_{12}$ might be thorough from $\mathcal{F}_1$ to some other frame. Notice that in Definition **??** we do not concern ourselves with the consequences of representations which are contradictory in $\mathcal{F}_2$.

The symbolism $\mathcal{S}_{AL}$ is thorough from $\mathcal{F}_A$ to $\mathcal{F}_{\text{dragon}}$. So $\mathcal{S}_{AL}$ is both thorough and proper from $\mathcal{F}_A$ to $\mathcal{F}_{\text{dragon}}$.

If P is a representation in $\mathcal{S}_{12}$ of $X \subset \mathcal{E}_1$, then we would like P to be contradictory in $\mathcal{F}_2$ if and only if X is contradictory in $\mathcal{F}_1$. Then we could say that the property of contradiction is preserved when we move from $\mathcal{F}_1$ to $\mathcal{F}_2$ using $\mathcal{S}_{12}$.

Definition 4.6 (Contradiction Preserving Symbolisms) If $\mathcal{F}_1 = (\mathcal{E}_1, \mathcal{L}_1, \mathcal{T}_1)$ and $\mathcal{F}_2 = (\mathcal{E}_2, \mathcal{L}_2, \mathcal{T}_2)$ are frames, and $\mathcal{S}_{12}$ is a symbolism from $\mathcal{F}_1$ to $\mathcal{F}_2$ whose domain is $\mathcal{D}_{12}$, then we say $\mathcal{S}_{12}$ is *contradiction preserving from $\mathcal{F}_1$ to $\mathcal{F}_2$* if and only if, for every $X \subset \mathcal{D}_{12}$, and every P which is a representation in $\mathcal{S}_{12}$ of X, we have P contradictory in $\mathcal{F}_2$ if and only if X is contradictory in $\mathcal{F}_1$.

No set of statements is contradictory in $\mathcal{F}_{\text{dragon}}$, so no representation in $\mathcal{S}_{AL}$ of $A \subset \mathcal{D}_{AL}$ will ever be contradictory in $\mathcal{F}_{\text{dragon}}$. But it is easy enough to find a set of ideas, $I \subset \mathcal{D}_{AL}$, which is contradictory in $\mathcal{F}_A$. Given that no representation in $\mathcal{S}_{AL}$ of I can be contradictory in $\mathcal{F}_{\text{dragon}}$, we see that $\mathcal{S}_{AL}$ cannot be contradiction preserving from $\mathcal{F}_A$ to $\mathcal{F}_{\text{dragon}}$.

Let P be a representation in $\mathcal{S}_{12}$ of $X \subset \mathcal{D}_{12}$. If $x \in \mathcal{D}_{12}$ is true in $\mathcal{F}_1$ with respect to X, then we would like every $p \in \mathcal{S}_{12}(x)$ to be true in $\mathcal{F}_2$ with respect to P. Conversly, we would also like to have $x \in \mathcal{D}_{12}$ true in $\mathcal{F}_1$ with respect to X whenever some $p \in \mathcal{S}_{12}(x)$ is true in $\mathcal{F}_2$ with respect to P. This is the case when $\mathcal{S}_{12}$ is 'truth preserving from $\mathcal{F}_1$ to $\mathcal{F}_2$'.

Definition 4.7 (Truth Preserving Symbolisms) If $\mathcal{F}_1 = (\mathcal{E}_1, \mathcal{L}_1, \mathcal{T}_1)$ and $\mathcal{F}_2 = (\mathcal{E}_2, \mathcal{L}_2, \mathcal{T}_2)$ are frames, and $\mathcal{S}_{12}$ is a symbolism from $\mathcal{F}_1$ to $\mathcal{F}_2$ whose domain is $\mathcal{D}_{12}$, then we say $\mathcal{S}_{12}$ is *truth preserving from* $\mathcal{F}_1$ *to* $\mathcal{F}_2$ if and only if, for every $X \subset \mathcal{D}_{12}$, every P a representation in $\mathcal{S}_{12}$ of X, and every $x \in \mathcal{D}_{12}$, $\mathcal{S}_{12}$ has the following properties:
(1) If x is true in $\mathcal{F}_1$ with respect to X, then every member of $\mathcal{S}_{12}(x)$ is true in $\mathcal{F}_2$ with respect to P.
(2) If a member of $\mathcal{S}_{12}(x)$ is true in $\mathcal{F}_2$ with respect to P, then x is true in $\mathcal{F}_1$ with respect to X.

We also have falsehood preserving symbolisms:

Definition 4.8 (Falsehood Preserving Symbolisms) If $\mathcal{F}_1 = (\mathcal{E}_1, \mathcal{L}_1, \mathcal{T}_1)$ and $\mathcal{F}_2 = (\mathcal{E}_2, \mathcal{L}_2, \mathcal{T}_2)$ are frames, and $\mathcal{S}_{12}$ is a symbolism from $\mathcal{F}_1$ to $\mathcal{F}_2$ whose domain is $\mathcal{D}_{12}$, then we say $\mathcal{S}_{12}$ is *falsehood preserving from* $\mathcal{F}_1$ *to* $\mathcal{F}_2$ if and only if, for every $X \subset \mathcal{D}_{12}$, every P a representation in $\mathcal{S}_{12}$ of X, and every $x \in \mathcal{D}_{12}$, $\mathcal{S}_{12}$ has the following properties:
(1) If x is false in $\mathcal{F}_1$ with respect to X, then every member of $\mathcal{S}_{12}(x)$ is false in $\mathcal{F}_2$ with respect to P.
(2) If a member of $\mathcal{S}_{12}(x)$ is false in $\mathcal{F}_2$ with respect to P, then x is false in $\mathcal{F}_1$ with respect to X.

It should be easy to convince yourself that $\mathcal{S}_{RL}$ is neither truth, falsehood, nor contradiction preserving from $\mathcal{F}_R$ to $\mathcal{F}_{\text{dragon}}$. A more interesting question is whether $\mathcal{S}_{RL}$ is truth preserving from $\mathcal{F}_R$ to $\mathcal{F}_L$.

A symbolism from $\mathcal{F}_1$ to $\mathcal{F}_2$ with all the properties defined above is said to be 'strict'.

Definition 4.9 (Strict Symbolisms) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are frames, and $\mathcal{S}_{12}$ is a symbolism from $\mathcal{F}_1$ to $\mathcal{F}_2$, then we say $\mathcal{S}_{12}$ is *strict from* $\mathcal{F}_1$ *to* $\mathcal{F}_2$ if and only if it is proper, thorough, contradiction preserving, truth preserving, and falsehood preserving from $\mathcal{F}_1$ to $\mathcal{F}_2$.

4.8 Symbolism of Events

Let us define a symbolism $\mathcal{S}_{UL}$ from $\mathcal{F}_U$ to $\mathcal{F}_L$. Let $\mathcal{D}_{UL}$, the domain of $\mathcal{S}_{UL}$, be the set of all events which you (the reader) think can be represented by statements in Restricted Mathematical English. We define $\mathcal{S}_{UL}$ to be such that for every event $x \in \mathcal{D}_{UL}$, we have $\mathcal{S}_{UL}(x) = \{p \mid p \in \mathcal{E}_L$ such that you think p represents $x\}$. If x is an event in $\mathcal{D}_{UL}$, then $\mathcal{S}_{UL}(x)$ is the set of all statements in $\mathcal{E}_L$ which you might use to represent x.

What use can $\mathcal{S}_{UL}$ be to us? Can it, somehow, allow us to represent the workings of the universe in a communicative frame, so that we can use the communicative frame to calculate the consequences of what we think are facts? For the sake of argument, let us suppose $\mathcal{S}_{UL}$ is strict from $\mathcal{F}_U$ to $\mathcal{F}_L$. We choose axioms in $\mathcal{E}_L$ which we think represent facts. If our basis set is contradictory in $\mathcal{F}_L$, then we know the set of events represented by our axioms must be contradictory in $\mathcal{F}_U$, for we have assumed $\mathcal{S}_{UL}$ is contradiction preserving from $\mathcal{F}_U$ to $\mathcal{F}_L$. By definition, no set of events contradictory in $\mathcal{F}_U$ can contain only facts, for if it is contradictory in $\mathcal{F}_U$, then it is impossible in our universe. Thus, if our axioms are contradictory in $\mathcal{F}_L$, then at least one of our axioms represents an event which is not a fact. On the other hand, if our axioms are not contradictory in $\mathcal{F}_L$, then we know the laws of the universe do not *preclude* the possibility of our being correct about the facts. If our axioms did indeed represent facts, then statements which were true in $\mathcal{F}_L$ with respect to our axioms would represent more facts. But is it the case that $\mathcal{S}_{UL}$ is strict from $\mathcal{F}_U$ to $\mathcal{F}_L$? Some people think it is. If you were a scientific externalist, you would believe $\mathcal{S}_{UL}$ to be strict from $\mathcal{F}_U$ to $\mathcal{F}_L$.

4.9 Final Remark

Frames and symbolisms do not pass judgement upon the things they describe. If we feel a symbolism from our mind to a communicative frame should, for example, be contradiction preserving, this does not mean the phrase 'contradiction preserving' passes any judgement upon the value of particular symbolisms. It means that *we* have passed judgement upon the value of what are *called* contradiction preserving symbolisms.

Chapter 5

Science

We begin this chapter with a quotation from Ludwig Wittgenstein.

'And we may not advance any kind of theory. There must not be anything hypothetical in our considerations. We must do away with all explanation and description alone must take its place. And this description gets its light, that is to say its purpose, from the philosophical problems. These are, of course, not empirical problems; they are solved, rather, by looking into the workings of our language, and that in such a way as to make us recognize those workings; in despite of an urge to misunderstand them. The problems are solved, not by giving new information, but by arranging what we have always known. Philosophy is battle against the bewitchment of our intelligence by means of language.'

This is paragraph 109 from Wittgenstein's last work, 'Philosophical Investigations'. It seems Wittgenstein had come to the conclusion that it was time philosophers stopped telling people what to believe, and instead concentrated upon finding ways of arranging what everyone already knew — or at least thought they knew. This is the purpose for which frames and symbolisms have been designed.

5.1 Effective Symbolism

We are taught to use language from childhood. This teaching is founded upon the equivalent alternative reply test. Once we are adult, we hope that we and the people we talk to shall attach similar meanings to the words we use. This would allow our communication to be accurate. If our communication is not accurate, what then? Would that be such a bad thing? Here is an argument in favour of insisting upon accurate communication:

'If our communication is not accurate, we shall neither understand nor be understood; and it is not as if this failure of understanding will be harmless. If our communication is embellished by our listener, this embellishment is all the more dangerous when it is at first insignificant, for it will then go unnoticed, to be added to by many other insignificant embellishments as our communication progresses. All of these embellishments are the creations of our listener's mind, and so are familiar to him, and do not clash with his intuition. If we are trying to communicate a new discovery of ours, a discovery we know to be counter-intuitive to our listener, then by the time we have finished descibing it, the idea he has in his mind will be so overwhelmed with additions of his own invention that, even if he agrees with us, he will never see the originality of our idea; and later, when he thinks of what we said, our idea will be lost in a tangle of his own embellishments. Without accurate communication it is not possible to communicate original thought. Inaccurate communication is fine, of course — essential even — if we want to preserve our beliefs behind

a cloud of ambiguity. But if we talk to explain, and listen to learn, then we must consider inaccurate communication to be a failing.'

Here is an argument against insisting upon accurate communication:

'When we are deprived of perfection, we must still make do. It may be easy to communicate accurately about mathematics and physics, but some things are beyond the reach of the cold clarity of mathematical language. We cannot express love and loyalty in terms of sets and functions. If we insist upon accuracy in our communication, then we would have to avoid talking about the human passions which, in the final analysis, make life worth living. It is better to try to communicate one's feelings — however vaguely — than it is to refuse to talk about them because they do not have the same clinical simplicity as the phenomena of gravitation and nuclear physics. Communication is the goal of communication, so if we can do a poor job of it that is better than doing no job at all.'

Both these arguments are examples of reasoning in which not only are the rules of logic used to move from one consequence to the next, but other tricks are used as well. In particular, we have the rule of 'rhetoric'. From the perspective of logic, rhetoric may be regarded as the addition of a new basis statement in the middle of a line of reasoning. If we picture the reasoner as someone hopping from stone to stone in a marsh, trying to get somewhere in particular, then the use of rhetoric is akin to his pulling a stepping stone out of his pocket, plopping it down, and using it to take another step. With no constraints placed upon our use of rhetoric, we are allowed to use as a stepping stone in our reasoning any statement which is not a paradox in $\mathcal{F}_L$. Paradoxes in $\mathcal{F}_L$ are forbidden out of tradition. With only the rules of logic, one is at the mercy of the stones which exist already in the marsh. With rhetoric, our reasoning is set free to wander where it chooses.

The most important steps in the italicised passages above are rhetorical. In the argument insisting upon accuracy we have:

'...these embellishments...do not clash with his intuition.'

In the counter-argument we have:

'...some things are beyond the reach of the cold clarity of mathematical language.'

If we can put down a stone down in the marsh wherever we like, then we can get anywhere in the marsh. It does not matter where we start off or what stones already exist. To get where we want to be, our only labour is pulling stepping stones out of our pockets.

Let us define a frame $\mathcal{F}_{\text{rhetoric}} = (\mathcal{E}_L, \mathcal{L}_L, \mathcal{T}_{\text{rhetoric}})$, where $\mathcal{T}_{\text{rhetoric}}(A \subset \mathcal{E}_L)$ is the set of all members of $\mathcal{E}_L$ which can be reached from the members of A using any combination of logic and rhetoric. It is easy to show that this frame is not particularly useful. Unconstrained rhetoric gives us too much freedom to prove things.

Let A be any subset of $\mathcal{E}_L$. Rhetoric allows us to use any statement in $\mathcal{E}_L$ as a stepping stone in our reasoning, so long as the statement is not a paradox in $\mathcal{F}_L$. Using unconstrained rhetoric I can assume, *'It is raining,'* is a consequence of my axioms. Later on in my reasoning I can assume, *'It is not raining,'* is a also consequence of my axioms. Then I can use the rule of conjunction to show that, *'It is raining. It is not raining,'* is a consequence of my axioms. Therefore, every $A \subset \mathcal{E}_L$ is contradictory in $\mathcal{F}_{\text{rhetoric}}$. The only way for $\mathcal{S}_{RL}$ to be contradiction preserving from $\mathcal{F}_R$ to $\mathcal{F}_{\text{rhetoric}}$ is for every set of ideas in the domain of $\mathcal{S}_{RL}$ to be contradictory in $\mathcal{F}_R$. Speaking for myself, I will say that $\mathcal{S}_{AL}$ is not contradiction preserving from $\mathcal{F}_A$ to $\mathcal{F}_{\text{rhetoric}}$. The frame $\mathcal{F}_{\text{rhetoric}}$ in conjunction with $\mathcal{S}_{AL}$ is not an effective model of the workings of my mind. It does not preserve contradiction, let alone truth or falsehood.

The same may be said for the use of $\mathcal{F}_{\text{rhetoric}}$ to model $\mathcal{F}_U$. Suppose we want to define a communicative frame in which to contemplate and discuss the workings of the universe. By definition, the events which occur in our universe are not contradictory in $\mathcal{F}_U$. But when we represent any set of events in $\mathcal{S}_{UL}$ we always find their representation to be contradictory in $\mathcal{F}_{\text{rhetoric}}$. Thus $\mathcal{F}_{\text{rhetoric}}$ does not lend itself to the representation of $\mathcal{F}_U$. If we for some philosophical reason of our own need a frame that models the workings of $\mathcal{F}_U$, then we must reject $\mathcal{F}_{\text{rhetoric}}$.

Having said all this about unrestrained rhetoric, we notice that a simple modification to $\mathcal{F}_{\text{rhetoric}}$ turns it into a frame which *is* an effective model of our minds. All we have to do is restrict the use of rhetoric in the frame so that it can be used only to introduce assumptions which we ourselves have already chosen to believe. Using rhetoric to introduce an assumption which we do not believe is forbidden in such a frame. To each of us there corresponds a frame of 'personalised rhetoric'. Each person's frame of personalised rhetoric acts as an effective model of his or her mind. The problem with frames of personalised rhetoric is that you cannot define them for other people without listing all the ideas that you believe. Listing your axioms is not enough, for when you use personalised rhetorical reasoning, you allow yourself to call upon all of your own beliefs. All the same, it is traditional for people to use frames of personalised rhetoric in arguments of philosophy and politics. Most people believe there to be a simple strict symbolism from $\mathcal{F}_U$ to their own frame of personalised rhetoric.

When a group of people is in disagreement, and they resolve to settle the matter in a communicative frame, it is unlikely the poeple in the group shall agree to select someone's frame of personalised rhetoric. Quite apart from the difficulty of defining such frames, everyone knows that everyone else can prove their point in their own frame of personalised rhetoric, so no one is going to agree to anyone else's frame. A communicative frame for resolving arguments should be easy to define and unbiased. The frame of Restricted Mathematical English and Logic is an example of such a frame.

There may be nothing to be gained from resolving our differences. Perhaps unending debate is better for us than subjecting our free spirits to something as constraining as logic. I do not want to express any opinion upon this matter. It is for each of us to decide what the criteria are for an effective symbolism. Each of us decides for ourselves what it is we want to acheive with reason and language.

5.2 Scientific Externalism

Scientific externalism is a branch of externalism. In you, my reader, scientific externalism would be the belief that $\mathcal{S}_{UL}$ is strict from $\mathcal{F}_U$ to $\mathcal{F}_L$. I say 'in you', because, if you recall, we defined $\mathcal{S}_{UL}$ in terms of your system of symbolising events with English statements. I can also define a symbolism from $\mathcal{F}_U$ to $\mathcal{F}_L$ using my own system of symbolising events. So can anyone else who speaks English. If p is a person, then let $\mathcal{S}^p_{UL}$ denote the symbolism from $\mathcal{F}_U$ to $\mathcal{F}_L$ defined in the same way for them as the original $\mathcal{S}_{UL}$ was defined for you. The original $\mathcal{S}_{UL}$ can be denoted $\mathcal{S}^R_{UL}$, where R is for reader, and my own symbolism of events would be $\mathcal{S}^A_{UL}$, where A is for author. We hope that all the years of applying the EAR test to our use of language will have made these symbolisms very similar.

With the above terminology, a person, p, is a scientific externalist if he believes that $\mathcal{S}^p_{UL}$ is strict from $\mathcal{F}_U$ to $\mathcal{F}_L$. Scientific externalists have every confidence that when they represent facts in $\mathcal{F}_L$, the consequences in $\mathcal{F}_L$ of the representation will correspond to more facts. Scientific externalists settle their disagreements in $\mathcal{F}_L$. They bow down to consequences in $\mathcal{F}_L$ because they believe these consequences represent facts. Ardent scientific externalists go so far as to believe that in order to learn the facts it is worth suffering shame and all manner of hardships.

5.3 The Dilemma of Prediction

Let us introduce Matilda, an ardent scientific externalist. Every morning so far in Matilda's life, the sun has risen in the morning. Tomorrow she plans to get up and see the sun rise personally. But, as an ardent scientific externalist, Matilda will not set her heart upon seeing the sun rise unless she can demonstrate in $\mathcal{F}_L$ that for the sun to rise tomorrow is a consequence of her experiences of the sun in the past. So she sits down and tries to reason it out.

She arranges her information in the following way. She decides she has exactly one trial of the sun's behaviour for each day of her life. She classifies the outcomes of these trials into two types. If the sun came up, then the result is of type 1. If the sun did not come up, then the result is of type 2. Each trail has a number to identify it: the number of the day in Matilda's life upon which the trial took place. It is now the n'th day in her life, so Matilda has a sequence of n outcomes. She proposes that there stretch befor her many more trials of the sun's behaviour. She imagines the sequence of trials going on until the N'th trial on the last day of her life. She does not yet know the outcomes of the $(n+1)$'th to N'th trials, but she hopes to be able to figure them out on the basis of the first n outcomes — which she does know.

Matilda has things arranged to her satisfaction, so she sets about trying to find out whether the sun will come up tomorrow. She has a sequence of N outcomes. The first n of these outcomes she believes are all of type 1, since she believes the sun has risen on every day of her life. What does logic have to say about the $(n+1)$'th outcome in the sequence, given that the first n outcomes are of type 1?

Matilda tries and tries, but she can get nothing from logic about the $(n+1)$'th outcome. Logic has absolutely nothing to say about the $(n+1)$'th outcome. Matilda is dismayed: what is the point in being a scientific externalist if one cannot use $\mathcal{F}_L$ to find out what is going to happen tomorrow?

It becomes obvious to Matilda that if she is to use her experience of the past to come to conclusions about the future, she shall have to make some assumptions about the bearing of the past upon the future. Being an ardent scientific externalist, Matilda is unwilling to start making such assumptions unless it is within the constraints of a strictly defined framework, where there is an impersonal system for determining which assumptions are to be allowed and which are not. This is what Empirical Science is designed to do. Whether it succeeds or not is a matter of opinion, but it is a very popular system all the same.

5.4 The Assumptions of Empirical Science

In order to define exactly what assumptions are made by empirical science, we must first find a standard form into which all experimental information may be transformed. Then the assumptions of empirical science may be defined in relation to this standard form.

We begin with Matilda's way of arranging her information. In the case of the sun rising, each trial had two possible outcomes. Let us consider some other types of trials. If we were rolling a die, then we could say there were six possible outcomes. We could also say there were only two possible outcomes: 'a six' or 'not a six'. If we were measuring the time it took an object to fall to earth from a height h, we could say there were two possible outcomes: either the time to fall is in keeping with gravity being an acceleration of 9.81 meters per second per second, or it is not. Whatever sequence of trails makes up our experimental information, we can always arrange the trial results so they are classified into one of m possible outcomes. What these possible outcomes are depends upon the way we intend to use our observations. If we are concerned only with the number of times a die comes up with a 6, and not with the relative occurances of numbers 1 to 5, then we would be inclined to classify our observations into 'a six' or 'not a six'. The first step toward transforming our information into standard form is to decide upon a set of possible outcomes. We call this set Υ. It has m members. We write $\Upsilon = \{\upsilon_1, \upsilon_2, \ldots, \upsilon_m\}$.

Matilda imagined a sequence of N trials of which she knew the outcomes of the first n. We can do something similar for any experiment. If we are rolling a die, then we have n trials whose outcomes are known, and we imagine N being the number of trials for which we shall ever know the outcomes.[1] Trials in the sequence do not have to be in chronological order. It could be that the first n are recent trials, and the next twenty are trials performed and recorded a hundred years ago. When we discover the results of these old trials, we can add them to our sequence as the $(n+1)$'th to $(n+20)$'th outcomes.

Let us regard the trials of our sequence as being ordered pairs (i, υ), where $i \in \mathbb{N}$ is the number of the trial, and $\upsilon \in \Upsilon$ is its outcome. The complete sequence of trials is the set of all N such ordered pairs. We denote this set Φ. If we let $I = \{i \mid i \in \mathbb{N} \text{ and } i \leq N\}$, then we see that Φ is a mapping from I to Υ, since Φ contains one and only one ordered pair (i, υ) for every $i \in I$. The set of ordered pairs representing the first n trials is denoted Ψ. We see that $\Psi \subset \Phi$. An 'experiment in standard empirical form' is an ordered triple (Υ, Φ, Ψ).

Experiments in Standard Empirical Form: Let Υ be a set with m members, where $m \in \mathbb{N}$. Let N be a member of $\mathbb{N}$. Let $I = \{i \mid i \in \mathbb{N} \text{ and } i \leq N\}$. Let Φ be a mapping from I to Υ. Let $n \in \mathbb{N}$ be such that $n \leq N$. Let $\Psi = \{(i, \upsilon) \mid i \leq n \text{ and } (i, \upsilon) \in \Phi\}$. We say $X = (\Upsilon, \Phi, \Psi)$ is an *experiment in standard empirical form*.

Now we equate some names with the components of an experiment in standard empirical form.

Empirical Terminology: Let $X = (\Upsilon, \Phi, \Psi)$ be an experiment in standard empirical form. We call Υ the *set of possible outcomes in X*. We call any $\upsilon \in \Upsilon$ a *possible outcome in X*. We call Φ the *set of trials in X*. We call any $\phi \in \Phi$ a *trial in X*. We call Ψ the *set of observed trials in X*. We call any $\psi \in \Psi$ an *observed trial in X*.

The assumption of empirical science, when applied to an experiment in standard empirical form, is called the 'assumption of adequate observation'. The assumption of adequate observation is this: that the proportion of observed trials which have a particular outcome is the same as the proportion of *all* trials with that outcome. For example, in Matilda's case, the assumption of adequate observation is that the proportion of type 1 outcomes in the n known trials is the same as the proportion of type 1 outcomes in all N trials. Applying the assumption of adequate observation to her experiment, Matilda can now conclude that every one of the N trials will have outcome 1 one the grounds that the first n trials all had outcome 1. Matilda has assumed that the outcomes of future trials are revealed by the outcomes of the observed trials. She assumes her observations are 'adequate'.

Before we come to the formal definition of the assumption of adequate observation, let us introduce some terminology to make the definition more eloquent.

Terminology for the Assumption of Adequate Observation: Let $X = (\Upsilon, \Phi, \Psi)$ be an experiment in standard empirical form. Let N be the number of members in Φ, and let n be the number of members in Ψ. For any $\upsilon \in \Upsilon$ let $n(\upsilon)$ denote the number of ordered pairs $(i, \upsilon) \in \Psi$, and let $N(\upsilon)$ denote the number of ordered pairs $(i, \upsilon) \in \Phi$. For any $\upsilon \in \Upsilon$, the *observed probability of υ in X*, denoted $p_X(\upsilon)$, is given by $p_X(\upsilon) = n(\upsilon)/n$. For any $\upsilon \in \Upsilon$ the *true probability of υ in X*, denoted $P_X(\upsilon)$, is given by $P_X(\upsilon) = N(\upsilon)/N$.

The number $P_X(\upsilon)$ is the fraction of all trials which have outcome υ, while $p_X(\upsilon)$ is the fraction of all observed trials which have outcome υ. The assumption of adequate observation is the assumption that for

[1] When I say 'know', I of course mean 'believe to be known', for, ultimately, everything we 'know' about the universe is known because we believe it to be known. It is an act of faith to say the external universe is there at all.

all $\upsilon \in \Upsilon$ we have $P_X(\upsilon) = p_X(\upsilon)$.

The Assumption of Adequate Observation: Let $X = (\Upsilon, \Phi, \Psi)$ be an experiment in standard empirical form. The *assumption of adequate observation in* X is equivalent to the assumption that for every $\upsilon \in \Upsilon$ we have the observed probability of υ in X equal to the true probability of υ in X.

Since all experiments can be resolved into standard empirical form, we see that all empirical assumptions can be resolved into the form of the assumption of adequate observation. Even the assumption that one's measuring instruments are accurate may be resolved into an assumption of adequate observation by arranging one's information about the instruments into an experiment in standard empirical form. Let us demonstrate further the use of the standard empirical form with another example.

5.5 An Experiment With Gravity

We introduce Matthew, an ardent scientific externalist. Matthew has been hired to do some fundamental research into gravitational attraction. His task is to predict how long it will take a golf ball to fall from a particular window to the street below. Unfortunately, he is not allowed to drop a golf ball from the window himself. The window is on the sixth floor. However, all the odd-numbered floors have a window in a public place which can be opened, so Matthew is able to drop golf balls late at night from all the odd-numbered floors. He also finds out how high above the ground is each of the windows.

When Matthew drops a golf ball from one of the windows, he always starts a stop watch. When he sees the ball hit the pavement, he stops the watch. He tries to measure the time it takes the golf balls to fall as accurately as possible, and believes his error in measurement to always be less than a quarter of a second. Having dropped a golf ball from each of the odd-numbered floors, Matthew takes his information home and hopes to apply the assumptions of empirical science so as to predict how long it will take the golf ball to fall from the sixth floor window.

Matthew plots his data on a graph. On the abscissa (the horizontal axis) he plots the height from which he dropped the golf ball, and on the mantissa (the vertical axis) he plots his measurement of the time taken for the ball to fall. He sees that he can draw a nice parabolic curve through his measurements. He could use this curve to make predictions about the time taken for objects to fall from even-numbered floors, but he wonders if he is allowed to draw such a curve. Who can say that the time taken for the golf ball to fall from the sixth floor will lie nicely between the time taken to fall from the fifth floor and the time taken to fall from the seventh floor? Why should he allow himself to draw a simple curve through his points, instead of a wiggley one? When it comes down to it, who is to say that one is entitled to draw a curve at all? Do golf balls always take the same amount of time to fall from the same height? If two golf balls are dropped from heights nearly the same, do they take nearly the same amount of time to fall?

To make matters more confusing, Matthew now rearranges his data with a mathematical transformation. He decides to plot on the mantissa the falling time with one second added whenever the floor number is even, and with no seconds added whenever the floor number is odd. In this new representation, the points of his observations, which are all from odd-numbered floors, are still in the same places on the graph. But when he draws a nice curve through them, this nice curve makes different predictions about how long golf balls take to fall than does the nice curve drawn through the points in the original representation. Intuitively, Matthew feels that the second representation is a silly one. But he also feels, as an ardent empirical scientist, that if the second representation really is silly, he should be able to come up with solid reasons for why it is so. Moreover, Matthew figures that if he can find these reasons, they will take him most of the way towards finding his answer.

After reflection, Matthew decides upon three further assumptions about gravity and golf balls. These assumptions are made in addition to his assumption that his recent measurements were accurate. The new assumptions are based upon Matthew's many years watching falling objects. They are the results of other experiments. His first assumption is that whenever one compares two trials in which golf balls are dropped from different floors, one will always find that the ball dropped from the higher floor takes longer to fall than the one dropped from the lower floor. Matthew has always observed this to be the case in the past, so he makes the assumption of adequate observation to assume it will also be the case in the future. Matthew's second assumption is that the *increase* in falling time which is observed when moving up one floor grows less as you go up floors. Thus, if the time taken to fall from the third floor is 1 second more than the time taken to fall from the second floor, then the time taken to fall from the fourth floor is less than one second more than the time taken to fall from the third floor. Matthew's third assumption is that if he were able to divide the height of the building into a very large number of tiny stories, and drop golf balls from all of them, the above two assumptions would still hold with respect to the new floors, regardless of how many new floors there were.

Returning to his original graph, Matthew finds himself in a position to draw a nice curve not simply because it is nice looking, but because the curve *must* be that way if it is to obey his assumptions. Firstly, on the grounds that he can divide the height of the building into any number of floors and still find his assumptions correct, he concludes that there must be a curve which passes through all his data points, and through all the heights in between them; this curve being such that its value measured on the mantissa at any height on the abscissa is equal to the time taken for a golf ball to fall from that height. Because of his first assumption, this curve can never slope downwards. It must be continuously rising with height, for Matthew has assumed it takes an object longer to fall from a higher floor. We can have as many floors as we like. So many, in fact, that there can be one arbitrarily close to any height we care to choose. Matthew's second assumption means that, although the curve is always sloping upwards, the steepness of this upward slope must always be decreasing as the height increases. Matthew is pleased to find these restrictions constrain the curve to lie in a narrow region on the graph. One curve satisfying the restrictions is given by $t = c\sqrt{h}$, where t is the time to fall, h is the height fallen, and c is a fixed number refferred to as 'a constant'.[2]

Now Matthew can measure the height of the sixth floor and make his prediction. He will not be able to make a perfect prediction, for there are a range of curves that fit his assumptions and his data. These curves lie close to together, so the predictions they make are also close together. If the predictions lie within half a second of eachother, then Matthew will be able to say, 'the time taken for the golf ball to fall from the sixth floor is within half a second of. . . ' The reason there are a range of curves fitting the data is that Matthew's measurements of time were uncertain by a quarter of a second. This means the curves need only pass within half a second of each of Matthew's measurements.

For the mathematicians among us, Matthew's assumptions imply the existence of a mapping, f, from the set of all heights to the set of all lengths of time, such that $f(h)$ is the time taken for a golf ball to fall from height h, and we always have $df(h)/dh > 0$ and $d^2f(h)/dh^2 < 0$. Matthew has also observed that $f(0) = 0$. This is enough to conclude that $f(h) = ch^n$ where $c > 0$ is a constant and $0 < n < 1$.

If Matthew were to represent his data in another way (such as the second way he thought up), then his assumptions would lead him to the same predictions. In his second representation, the curve would wiggle up and down in order to obey the assumptions. Subtracting one second form this wiggley curve at even floors, and subtracting no seconds on odd floors, would give the right answer.

In terms of the standard empirical form, Matthew used his three assumptions to justify a particular arrangement of his data. Then he made the assumption of adequate observation in his chosen arrangement. In a trial when a golf ball is dropped from a height h and takes a time t to fall, Matthew regards there to be two outcomes: either t is within half a second of $c\sqrt{h}$, or not. This arrangement of the data was

[2]The expression '$t = c\sqrt{h}$' means 'the time to fall is equal to a constant times the square root of the height fallen.'

chosen on the grounds of Matthew's three assumptions. These assumptions are themselves instances of the assumption of adequate observation. Matthew noticed that the way he chose to represent his data would affect the implications of his assumption of adequate observation. He needed to call upon other assumptions of adequate observation in order to decide upon the outcomes he would use in the standard empirical representation of his data. It may be that those other assumptions of adequate observation were themselves based upon still more assumptions of adequate observation. We can regard the work of the empirical scientist as that of generating new assumptions of adequate observation either out of nothing (rarely), or with the help of existing assumptions of adequate observation (far more common).

5.6 The Arbitrary Nature of Things

We have defined a standard empirical form for experiments, and we have defined the empirical scientist's assumption, which is the assumption of adequate observation. Every experiment may be transformed into standard empirical form. We find it is all too easy to perform this transformation. For any set of observations, there are not just one, but many — if not an infinity — of ways we can express the observations in standard empirical form. When we want to decide upon the bearing our observations have upon the outcomes of future trials, we must decide upon a way to represent those observations. Different representations, when combined with the assumption of adequate observation, give us different predictions. Fortunately, we are usually able to use existing (and therefore, we assume, trustworthy) assumptions of adequate observation to force upon us the choice of a particular representation of our observations. In the case of Matthew's gravitational experiment, he was able to use existing assumptions of adequate observation to narrow down the possible representations. The representations which were acceptable, when combined with the assumption of adequate observation, led to predictions which all lay within half a second of one another.

Much of the time, our choice of representation will not be arbitrary because we have other assumptions of adequate observation to help us. But we must enquire about those other assumptions of adequate observation. Are they made arbitrarily? If not, then how about the further assumptions of adequate observation upon which they must be based? Eventually we are sure to find an assumption of adequate observation for which the representation of our observations has not been chosen on the grounds of other assumptions of adequate observation. In such a case, is there any other way we can decide upon a choice of representation?

Let us consider again Matilda's experiment with the sun. She chose to represent her trials with the outcomes 'the sun came up' and 'the sun did not come up'. That seems sensible. Is there any way of showing in $\mathcal{F}_L$ that this representation is the most basic, the most simple, and therefore the most sensible? Let us explore the matter.

First we represent the outcome 'the sun came up' with the symbol A, and the outcome 'the sun did not come up' with the symbol $\overline{A}$. Let us consider an alternative representation of the data. We define two new outcomes. The first outcome, which we shall denote B, occurs on the i'th trial if for that trial we have 'either A or $i > n$, but not both'. The second of our two new outcomes, which we shall denote $\overline{B}$, occurs whenever B does not occur. If the sun comes up and $i \leq n$, then outcome B occurs. If the sun does not come up and $i \leq n$, then outcome $\overline{B}$ occurs. If the sun comes up and $i > n$, then outcome $\overline{B}$ occurs. If the sun does not come up and $i > n$, then outcome B occurs.

The 'either one or the other but not both' expression is called 'exclusive or'. We denote it $\otimes$. Thus outcome B is the same as '$A \otimes (i > n)$'. I introduce this notation for brevity.

Representing Matilda's observations in terms of the outcomes B and $\overline{B}$, we see that all n observed trials have outcome B. The assumption of adequate observation applied to this representation gives us the prediction that trial $n+1$ will also have outcome B. This means that on trial $n+1$ we have *either A or*

$i > n$, but not both. On trial $n+1$ we have $i > n$, so for B to occur (and our assumption of adequate observation says it *will* occur), we cannot have A occurring. Thus A does not occur on the $(n+1)$'th trial, so the sun does not come up the next morning according to this empirically scientific reasoning.

We are suspicious, perhaps, of the business with the $i > n$ in the definition of outcomes B and $\overline{B}$, but let us define two more outcomes and see what happens. Let outcome C be the same as '$B \otimes (i > n)$', and let outcome $\overline{C}$ occur whenever C does not. For the first n trials we have B occuring and $i \leq n$, so outcome C occurs for all trials with $i \leq n$. The assumption of adequate observation applied to this representation states that C will occur on the $(n+1)$'th trial. What does that mean? To find out what it means for outcome C to occur, let us consider in turn the four possible combinations of A and $i > n$.

(1) If A and $i > n$ occur, then B does not occur, so C occurs.
(2) If A occurs while $i > n$ does not occur, then B occurs, so C occurs.
(3) If A does not occur while $i > n$ does occur, then B occurs, so C does not occur.
(4) If neither A nor $i > n$ occur, then B does not occur, so C does not occur.

Examining these four cases, we see that C occurs if and only if A occurs, so C is exactly equivalent to A. Thus we have A the same as $B \otimes (i > n)$, and B the same as $A \otimes (i > n)$. There is perfect duality between A and B.

As far as any logical analysis in $\mathcal{F}_L$ is concerned, there is no way we can say the outcomes A and $\overline{A}$ are any more honest than the outcomes B and $\overline{B}$. We cannot show in $\mathcal{F}_L$ that A is independent of i while B is not. We might hope to do so in order to argue that A is better by virtue of its independence of i. But as far as logic is concerned, A is no more independent of i than B. We cannot turn to $\mathcal{F}_L$ to settle the matter of which representation to choose. Logical dualities of the type encountered above can always be found. Instead of i as the source of our dilemmas, we could use the date upon which the trial takes place, or any number of other criteria.

Once again, Matilda is dismayed. It seems that empirical scientists make a habit of using the first representation of their data which springs to mind. They regard this representation as the best, for it is to them the simplest. The equating of 'best' with 'simplest' is called the principal of 'Occam's Razor'. With Occam's razor we cut away all uneccessary embellishments of our representation — or rather, we cut away what *we* see as uneccessary embellishments of our representation. Thus we would dismiss outcomes B and $\overline{B}$ in favour of A and $\overline{A}$, the latter two being to our minds simpler and more direct.

Matilda decides that the only thing for her to do now is to accept Occam's razor without further question. Whenever she must represent observations in standard empirical form, she will choose the representation her gut intuition tells her is the simplest and most direct. In the past she regarded empirical science as a pure and impersonal way of generating predictions. Now she regards it more as an aid to keeping a clear and complete record of her assumptions. This, she believes, is the best she can do in the face of the arbitrary nature of things.

5.7 Bold Scientific Externalists

Let $\mathcal{S}^p_{UL}$ be the symbolism used by a particular scientific externalist to represent events with statements in $\mathcal{E}_L$. We also have $\mathcal{D}^p_{UL}$ representing the domain of $\mathcal{S}^p_{UL}$, and $\mathcal{G}^p_{UL}$ representing the symbol set of $\mathcal{S}^p_{UL}$. Let $E \subset \mathcal{D}^p_{UL}$, and let P be any representation of E in $\mathcal{S}^p_{UL}$.

Our scientific externalist assumes $\mathcal{S}^p_{UL}$ is strict from $\mathcal{F}_U$ to $\mathcal{F}_L$. This means he assumes the set E is contradictory in $\mathcal{F}_U$ if and only if P is contradictory in $\mathcal{F}_L$. If P is contradictory in $\mathcal{F}_L$, then he assumes E is an impossible set of events. Let us suppose E is not contradictory in $\mathcal{F}_U$. Then the scientist assumes P is not contradictory in $\mathcal{F}_L$. Our scientist has assumed $\mathcal{S}^p_{UL}$ is thorough from $\mathcal{F}_U$ to $\mathcal{F}_L$ (part of its being strict), which means he assumes $\mathcal{T}_L(P) \subset \mathcal{G}^p_{UL}$. That is, he assumes every consequence in $\mathcal{F}_L$ of P represent an event in the domain of $\mathcal{S}^p_{UL}$. He has assumed $\mathcal{S}^p_{UL}$ to be truth preserving from $\mathcal{F}_U$ to $\mathcal{F}_L$, so he assumes all consequences in $\mathcal{F}_L$ of P represent conseqeunces in $\mathcal{F}_U$ of E.

We see that *if* $\mathcal{S}^p_{UL}$ is strict, then our scientist gains a powerful tool for modelling the workings of the universe. Saying $\mathcal{S}^p_{UL}$ is strict may seem bold, but it is what every scientific externalist does.

So far we have made an unspoken assumption that $\mathcal{D}^p_{UL}$ does *not* include all events, and that $\mathcal{G}^p_{UL}$ does *not* contain all statements in $\mathcal{E}_L$. Earlier this century, a school of philosophy called 'logical atomism' was popular. The logical atomists believed it was possible to create a perfect language to represent all events, and that logic applied to events represented in the perfect language would give us a perfect model of the world. The logical atomists constructed their perfect languages out of simple statements called 'atomic statements.' Let $\mathcal{F}_P = (\mathcal{E}_P, \mathcal{L}_P, \mathcal{T}_P)$ represent a frame such as the logical atomists would use to model the world. The domain of $\mathcal{F}_P$ is a language of atomic statements. The consequence mapping of $\mathcal{F}_P$ is defined by a formal system of logic in the same way $\mathcal{T}_L$ is defined by the rules of Chapter 2. The logical atomists assumed that every event is represented by a unique atomic statement, and that every atomic statement represents a unique event. Let $\mathcal{S}_{UP}$ be the symbolism representing a logical atomist's use of atomic statements to represent events. The logical atomists believed $\mathcal{S}_{UP}$ to be strict from $\mathcal{F}_U$ to $\mathcal{F}_P$. They believed the domain of $\mathcal{S}_{UP}$ was $\mathcal{E}_U$ and that its symbol set was $\mathcal{E}_P$.

Logical atomism lost popularity because of the work of a mathematician called Goedel. Goedel showed that even in a language of atomic statements it was possible to phrase statements of the form, *'This statement is not a theorem.'* He then argued that such a statement can neither be proved nor disproved. If, *'This statement is not a theorem,'* is a theorem, then it implies that it is not a theorem. If, *'This statement is not a theorem,'* is not a theorem, then we see that it is a theorem. And so, Goedel argued, it is a fact that the statement, *'This statement is not a theorem,'* is not a theorem. Thus, *'This statement is not a theorem,'* represents a fact, and yet is not a theorem in our logical system. Goedel concluded that logical truth was not fact, so logical atomism could not work.

In terms of the symbolism $\mathcal{S}_{UP}$, Goedel showed $\mathcal{S}_{UP}$ cannot be truth preserving from $\mathcal{F}_U$ to $\mathcal{F}_P$. He did this by showing that the symbol set of $\mathcal{S}_{UP}$ includes statements whose behaviour in the logical system is not in keeping with the behaviour of the events they represent. To make his point, Goedel used statements of the form, *'This statement is not a theorem.'* The point can also be made with statements of the form, *'The event represented by this statement is inherently impossible.'* Such statements are not paradoxes in $\mathcal{F}_P$, but they are paradoxes in $\mathcal{F}_U$, so we show $\mathcal{S}_{UP}$ is not contradiction preserving from $\mathcal{F}_U$ to $\mathcal{F}_P$. We excluded from the language of Restricted Mathematical English all statements with meanings akin to, *'This statement is not a theorem.'* However, we did not exclude statements with meanings akin to, *'The event represented by this statement is inherently impossible.'* Therefore, if the symbol set of $\mathcal{S}^p_{UL}$ contains statements with meanings akin to, *'The event represented by this statement is inherently impossible,'* then it is impossible for $\mathcal{S}^p_{UL}$ to be strict from $\mathcal{F}_U$ to $\mathcal{F}_L$.

The logical atomists ran into problems because the domain of their symbolism was the set of *all* events, and its symbol set was the set of *all* atomic statements. We can restrict the domain of $\mathcal{S}^p_{UL}$ so that it contains only events which lend themselves to representation in logic. Then it is possible that $\mathcal{S}^p_{UL}$ is strict from $\mathcal{F}_U$ to $\mathcal{F}_L$. Certainly no one has come up with any logical reason for why $\mathcal{S}^p_{UL}$ cannot be strict.

5.8 Final Remark

In order to get more of what we want out of a world which does not bow to our beliefs, we must use a system of reasoning in which consequences arise out of experiences according to the same patterns by which events follow one another in the world around us. Scientists think that logical systems of reasoning can provide us with accurate models of the workings of the universe. But it is not true to say that scientists come to their conclusions soley through an application of logic to the bare facts of their observations. In this chapter we found that in order to make predictions about the future on the basis of their observations of the past, scientists must make many assumptions about the bearing of the past upon the future. These assumptions have no foundation in logic. However, scientists would argue that the fruits of science are justification enough for scientific beliefs.